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Saint-Venant's problem for Cosserat shells with voids

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Abstract

In this paper we investigate the mechanical behavior of Cosserat shells made from a material with voids. We formulate Saint-Venant's problem for cylindrical shells and determine the solution of the relaxed problem. Then, we apply the theoretical results to study the deformation of circular cylindrical shells. We also compare the solution of Saint-Venant's problem for Cosserat shells with corresponding results from the classical theory of shells.

Keywords: Cylindrical Cosserat shells; Elastic materials with voids; Extension; Bending; Torsion; Flexure

1. Introduction

The torsion of elastic cylindrical shells modelled as Cosserat surfaces has been studied by Wenner (1968). A Cosserat surface is a two-dimensional continuum with a deformable vector (called *director*) assigned to every point of the surface. For a detailed presentation of the theory of shells described as Cosserat surfaces we refer to the monograph of Naghdi (1972). In the last decades, the theory of Cosserat shells has received considerable attention and has been investigated by many scientists (see e.g., Steele, 1971; Rubin, 1987; Antman, 1995; Steigmann, 1999). In the monograph of Rubin (2000) several applications of the Cosserat theories are described. An interesting application of Cosserat shells for the modelling of interphases in elastic media has recently been presented by Rubin and Benveniste (2004).

In the present paper, we extend the results of Wenner (1968) and consider the deformation of Cosserat cylindrical shells made from a material with *voids* (also called *pores*). Moreover, in addition to the torsion problem, we also investigate the extension, bending and flexure problems for porous cylindrical shells. In the case when the porosity is zero, we find the same solution for the torsion problem as the one obtained by

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Wenner, with the help of a different method. We mention that the solutions of the extension, bending and flexure problems for the case of Cosserat shells which are not porous have been deduced previously by Bîrsan (2004).

For our purpose, we employ the theory established by Nunziato and Cowin (1979) and Cowin and Nunziato (1983) for elastic materials with voids. In this theory, the bulk density is written as the product of two fields: the matrix material density field and the volume fraction field. Thus, an additional degree of kinematical freedom is introduced. Capriz and Podio-Guidugli (1981) have shown that the Nunziato–Cowin theory for elastic materials with voids can also be regarded as a particular case of the theory of media with microstructure. In the last twenty years, there has been much written on the subject of elastic materials with voids (see e.g., Capriz, 1989; Ciarletta and Ieşan, 1993). Several results concerning the theory of Cosserat shells made from a material with voids have been established by Bîrsan (2000). The Nunziato–Cowin theory was also employed to investigate the bending of thermoelastic porous plates in Bîrsan (2003).

The first part of this article contains a review of the basic equations that govern the equilibrium of Cosserat shells made from an isotropic material with voids, in the context of the linear theory. Then, we formulate Saint-Venant's problem for cylindrical shells. In Section 4 we determine the solution of the relaxed Saint-Venant's problem for the case of open cylindrical surfaces. To this aim, we employ a method established by Ieşan (1986, 1987) in the context of the classical theory of elasticity and we separate the relaxed problem in two parts: the extension-bending-torsion problem and the flexure problem. In Section 5 we study the corresponding deformations of closed cylindrical shells. Then, we use the theoretical results to determine the solution for circular cylindrical shells. In the last section, we compare the solution of Saint-Venant's problem for Cosserat shells (in the non-porous case) with the corresponding results from the classical theories of shells and plates. Also, we observe an interesting analogy with the classical Saint-Venant's solution from the three-dimensional theory of elasticity.

2. Basic equations

Naghdi (1972) has discussed the theory of shells modelled as Cosserat surfaces. The theory of thermoelastic Cosserat shells with voids was presented by Bîrsan (2000).

In the present paper we confine our attention to the isothermal linear theory for Cosserat shells with voids and we begin with a review of the basic equations. We mention that this theory is *exact*, in the sense that it involves *no approximations*, beyond those already assumed by the linearity of the theory.

2.1. Cosserat shells with voids

Let $\mathscr{S}$ be the reference configuration of a Cosserat surface. We consider a curvilinear material coordinate system θ^α $(\alpha = 1, 2)$ on $\mathscr{S}$ and assume that $(\theta^1, \theta^2) \in \Sigma$, where Σ is an open bounded set of $\mathbb{R}^2$. The surface $\mathscr{S}$ is defined by an injective mapping $\mathbf{R}$ of class C^2 from $\overline{\Sigma}$ into the Euclidean three-dimensional space. We denote by $\boldsymbol{r}(\theta^\alpha, t)$ the position vector and by $\boldsymbol{d}(\theta^\alpha, t)$ the deformable director assigned to the material point of the surface $\mathscr{S}$ which coordinates are (θ^α) at time t. The motion of the Cosserat surface is defined by

$$\boldsymbol{r} = \boldsymbol{r}(\theta^\alpha, t), \quad \boldsymbol{d} = \boldsymbol{d}(\theta^\alpha, t), \quad (\theta^\alpha) \in \Sigma, \; t \in \mathscr{T}, \tag{2.1}$$

where $\mathscr{T}$ is a time interval. Let $\boldsymbol{R}(\theta^\alpha)$ and $\boldsymbol{D}(\theta^\alpha)$ be the position vector and the deformable director, respectively, in the reference configuration.

We consider the covariant base vectors along the θ^α-curves and the unit normal to the surface $\mathscr{S}$ defined by

$$\boldsymbol{A}_\alpha = \frac{\partial \boldsymbol{R}}{\partial \theta^\alpha} \quad (\alpha = 1, 2), \quad \boldsymbol{A}_3 = \frac{\boldsymbol{A}_1 \times \boldsymbol{A}_2}{| \boldsymbol{A}_1 \times \boldsymbol{A}_2 |}. \tag{2.2}$$

Let us denote by $A_{\alpha\beta}$ and $B_{\alpha\beta}$ the first and second fundamental forms of the surface $\mathscr{S}$

$$A_{\alpha\beta} = \boldsymbol{A}_\alpha \cdot \boldsymbol{A}_\beta, \quad B_{\alpha\beta} = B_{\beta\alpha} = -\boldsymbol{A}_\alpha \cdot \boldsymbol{A}_{3,\beta} = \boldsymbol{A}_3 \cdot \boldsymbol{A}_{\alpha,\beta},$$

where a subscript comma represents partial differentiation with respect to the surface coordinates (θ^α). Throughout this paper we employ the usual summation convention. The Latin subscripts are understood to range over the integers 1, 2, 3, whereas Greek subscripts are confined to the range 1, 2.

We remind that in the Nunziato–Cowin theory of materials with voids the mass density ρ has the decomposition

$$\rho = \eta\gamma,$$

where γ is the density field of the matrix material and η is the volume fraction field ($0 < \eta \leqslant 1$). In the reference configuration we have $\rho_0 = \eta_0\gamma_0$.

The infinitesimal displacement $\boldsymbol{u}$, the director displacement $\boldsymbol{\delta}$ and the change in the volume fraction field φ are defined by

$$\boldsymbol{u} = \boldsymbol{r} - \boldsymbol{R}, \quad \boldsymbol{\delta} = \boldsymbol{d} - \boldsymbol{D}, \quad \varphi = \eta - \eta_0.$$

We designate by u_i and δ_i the components $u_i = \boldsymbol{u} \cdot \boldsymbol{A}_i$, $\delta_i = \boldsymbol{\delta} \cdot \boldsymbol{A}_i$.

We confine our attention to elastic porous shells with constant thickness in the reference configuration. According to Naghdi (1972,p. 447), this class of shells is characterized by the fact that the reference director coincides with the unit normal to the reference surface, i.e. $\boldsymbol{D} = \boldsymbol{A}_3$.

The linear strain measures $e_{\alpha\beta}$, γ_i and $\rho_{i\alpha}$ satisfy the following geometrical equations:

$$\begin{aligned} &e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - B_{\alpha\beta}u_3, \quad \gamma_\alpha = \delta_\alpha + u_{3,\alpha} + B^\beta_\alpha u_\beta, \\ &\gamma_3 = \delta_3, \quad \rho_{\beta\alpha} = \delta_{\beta|\alpha} - B^\gamma_\alpha u_{\gamma|\beta} + B^\gamma_\alpha B_{\beta\gamma}u_3, \quad \rho_{3\alpha} = \delta_{3,\alpha}, \end{aligned} \tag{2.3}$$

where a subscript vertical bar stands for covariant differentiation with respect to the metric tensor $A_{\alpha\beta}$.

Let c be an arbitrary curve on $\mathscr{S}$ and let $\boldsymbol{N}$, $\boldsymbol{M}$ and h be the contact force, the contact director couple and the equilibrated stress, respectively, acting per unit length of c (see Naghdi, 1972). Then, we have the following relations of Cauchy type:

$$\boldsymbol{N} = \boldsymbol{N}^\alpha \nu_\alpha, \quad \boldsymbol{M} = \boldsymbol{M}^\alpha \nu_\alpha, \quad h = h^\alpha \nu_\alpha, \tag{2.4}$$

where $\boldsymbol{\nu} = \nu_\alpha \boldsymbol{A}^\alpha$ represents the unit normal to c tangent to the surface $\mathscr{S}$.

We define $N^{\alpha\beta}, V^\alpha$ and $M^{\alpha i}$ by the relations

$$\boldsymbol{N}^\alpha = N^{\alpha\beta}\boldsymbol{A}_\beta + V^\alpha \boldsymbol{A}_3, \quad \boldsymbol{M}^\alpha = M^{\alpha i}\boldsymbol{A}_i \tag{2.5}$$

and we introduce the notation $N'^{\alpha\beta}$ for the following expression:

$$N'^{\alpha\beta} = N^{\alpha\beta} + B^\beta_\gamma M^{\gamma\alpha}. \tag{2.6}$$

The equations of equilibrium for porous Cosserat surfaces are

$$N^{\alpha\beta}{}_{|\alpha} - B^\beta_\alpha V^\alpha + \rho_0 f^\beta = 0, \quad V^\alpha{}_{|\alpha} + B_{\alpha\beta}N^{\alpha\beta} + \rho_0 f^3 = 0, \tag{2.7}$$

$$M^{\alpha\beta}{}_{|\alpha} - V^\beta + \rho_0 l^\beta = 0, \quad M^{\alpha 3}{}_{|\alpha} - V^3 + \rho_0 l^3 = 0, \tag{2.8}$$

$$h^\alpha{}_{|\alpha} - g + \rho_0 p = 0. \tag{2.9}$$

In the relations (2.7), (2.8) the field quantities per unit mass f^i and l^i stand for the components of the assigned force and assigned director couple, respectively. In the equation of equilibrated force (2.9), g represents the intrinsic equilibrated body force per unit area and p is the external equilibrated body force per unit mass.

The constitutive equations for homogeneous porous Cosserat shells possessing holohedral isotropy (i.e. isotropy with a centre of symmetry, see Naghdi, 1972, Section 13γ) are the following:

$$\begin{aligned} N'^{\alpha\beta} &= [\alpha_1 A^{\alpha\beta}A^{\gamma\delta} + \alpha_2(A^{\alpha\gamma}A^{\beta\delta} + A^{\alpha\delta}A^{\beta\gamma})]e_{\gamma\delta} + \alpha_9 A^{\alpha\beta}\gamma_3 + \beta_4 A^{\alpha\beta}\varphi, \\ V^{\alpha} &= \alpha_3 A^{\alpha\beta}\gamma_\beta, \quad V^3 = \alpha_4\gamma_3 + \alpha_9 A^{\alpha\beta}e_{\alpha\beta} + \beta_5\varphi, \\ M^{\beta\gamma} &= (\alpha_5 A^{\alpha\beta}A^{\gamma\delta} + \alpha_6 A^{\alpha\gamma}A^{\beta\delta} + \alpha_7 A^{\alpha\delta}A^{\beta\gamma})\rho_{\gamma\delta}, \\ M^{\alpha 3} &= \alpha_8 A^{\alpha\beta}\rho_{3\beta} + \beta_2 A^{\alpha\beta}\varphi_{,\beta}, \\ g &= \beta_3\varphi + \beta_4 A^{\alpha\beta}e_{\alpha\beta} + \beta_5\gamma_3, \quad h^{\alpha} = \beta_1 A^{\alpha\beta}\varphi_{,\beta} + \beta_2 A^{\alpha\beta}\rho_{3\beta}, \end{aligned} \tag{2.10}$$

where $\alpha_1,\ldots,\alpha_9$ and $\beta_1,\ldots,\beta_5$ are constant constitutive coefficients.

In this paper, we consider static deformations of homogeneous and holohedral isotropic Cosserat shells subject to boundary conditions of the form

$$\boldsymbol{N} = \widetilde{\boldsymbol{N}}, \quad \boldsymbol{M} = \widetilde{\boldsymbol{M}}, \quad h = \widetilde{h} \quad \text{on } \partial\mathscr{S}, \tag{2.11}$$

where $\widetilde{\boldsymbol{N}}, \widetilde{\boldsymbol{M}}$ and $\widetilde{h}$ are prescribed functions.

2.2. Cylindrical Cosserat surfaces

In this section we deduce the basic equations for cylindrical Cosserat surfaces made from a material with voids.

Let us assume that the reference configuration $\mathscr{S}$ of a Cosserat shell is a cylindrical surface. We consider a rectangular Cartesian coordinate frame $Ox_1x_2x_3$ such that the generator of the surface $\mathscr{S}$ is parallel to Ox_3 and $\mathscr{S}$ is situated between the planes $x_3 = 0$ and $\bar{h}$. We denote by $\mathscr{C}_z$ the curve section of the surface perpendicular to the generator, lying in the plane $x_3 = z$. On the surface $\mathscr{S}$, we choose the curvilinear coordinates $\theta^1 = s$, $\theta^2 = z$, where $s \in [0,\bar{s}]$ is the arc parameter along the curve $\mathscr{C}_z$ and $z = x_3$, with $z \in [0,\bar{h}]$. The parametric equation of the surface $\mathscr{S}$ is

$$\boldsymbol{R} = \boldsymbol{R}(s,z) = x_\alpha(s)\boldsymbol{e}_\alpha + z\boldsymbol{e}_3, \qquad s \in [0,\bar{s}], \quad z \in [0,\bar{h}], \tag{2.12}$$

where $\boldsymbol{e}_i$ represent the unit vectors along the axes Ox_i. The curves $\mathscr{C}_z$ are assumed to be simple (open or closed) curves of class C^3. The unit tangent and normal vectors to $\mathscr{C}_z$ are given by

$$\boldsymbol{\tau}(s) = x'_\alpha(s)\boldsymbol{e}_\alpha, \quad \boldsymbol{n}(s) = \epsilon_{\alpha\beta}x'_\beta(s)\boldsymbol{e}_\alpha, \tag{2.13}$$

where $\epsilon_{\alpha\beta}$ is the two-dimensional alternator defined by $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{11} = \epsilon_{22} = 0$ and we use the notation $(\ )' = \mathrm{d}(\ )/\mathrm{d}s$. The following relations take place:

$$\begin{aligned} &\boldsymbol{A}_1 = \boldsymbol{\tau}, \quad \boldsymbol{A}_2 = \boldsymbol{e}_3, \quad \boldsymbol{A}_3 = \boldsymbol{n}, \\ &x''_\alpha(s) = \epsilon_{\beta\alpha}x'_\beta(s)\sigma'(s), \quad \sigma'(s) = \frac{1}{R(s)} = \epsilon_{\alpha\beta}x'_\alpha(s)x''_\beta(s), \\ &A_{\alpha\beta} = \delta_{\alpha\beta}, \quad B_{11} = -\frac{1}{R(s)}, \quad B_{12} = B_{21} = B_{22} = 0, \end{aligned} \tag{2.14}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, $R(s)$ denotes the curvature radius of $\mathscr{C}_z$ and $\sigma(s)$ designates the angle between the vectors $\boldsymbol{\tau}(s)$ and $\boldsymbol{e}_1$. We observe that the Christoffel symbols associated with the surface $\mathscr{S}$ are all zero: $\Gamma^\lambda_{\alpha\beta} = 0$. Thus, the physical components of any tensor coincide with the covariant and with the contravariant components of the same tensor (see Naghdi, 1972, Section A.4). Taking into account that $\theta^1 = s$, $\theta^2 = z$ and $\boldsymbol{A}_3 = \boldsymbol{n}$, in what follows we shall write the subscripts s, z and n instead of the indices 1, 2 and 3, respectively, for the components u_i, δ_i, $e_{\alpha\beta}$, γ_i, $\rho_{i\alpha}$, $N^{\alpha\beta}$, V^i, $M^{\alpha i}$ and h^α of the tensors defined in Section 2.1.

Then, the geometrical relations (2.3) become

$$
\begin{aligned}
& e_{ss} = \frac{\partial u_s}{\partial s} + \frac{u_n}{R(s)}, \quad e_{zz} = \frac{\partial u_z}{\partial z}, \quad e_{sz} = e_{zs} = \frac{1}{2}\left(\frac{\partial u_s}{\partial z} + \frac{\partial u_z}{\partial s}\right),\\
& \gamma_s = \delta_s - \frac{u_s}{R(s)} + \frac{\partial u_n}{\partial s}, \quad \gamma_z = \delta_z + \frac{\partial u_n}{\partial z}, \quad \gamma_n = \delta_n,\\
& \rho_{ss} = \frac{\partial \delta_s}{\partial s} + \frac{1}{R(s)}\frac{\partial u_s}{\partial s} + \frac{u_n}{R^2(s)}, \quad \rho_{zz} = \frac{\partial \delta_z}{\partial z}, \quad \rho_{sz} = \frac{\partial \delta_s}{\partial z},\\
& \rho_{zs} = \frac{\partial \delta_z}{\partial s} + \frac{1}{R(s)}\frac{\partial u_s}{\partial z}, \quad \rho_{ns} = \frac{\partial \delta_n}{\partial s}, \quad \rho_{nz} = \frac{\partial \delta_n}{\partial z}.
\end{aligned} \tag{2.15}
$$

In view of the constitutive equations (2.10), in the case of cylindrical Cosserat shells we obtain

$$
\begin{aligned}
& N_{ss} = (\alpha_1 + 2\alpha_2)e_{ss} + \alpha_1 e_{zz} + \frac{\alpha_5 + \alpha_6 + \alpha_7}{R(s)}\rho_{ss} + \frac{\alpha_5}{R(s)}\rho_{zz} + \alpha_9\gamma_n + \beta_4\varphi,\\
& N_{zz} = \alpha_1 e_{ss} + (\alpha_1 + 2\alpha_2)e_{zz} + \alpha_9\gamma_n + \beta_4\varphi,\\
& N_{sz} = 2\alpha_2 e_{sz}, \quad N_{zs} = 2\alpha_2 e_{sz} + \frac{1}{R(s)}(\alpha_6\rho_{zs} + \alpha_7\rho_{sz}),\\
& V_s = \alpha_3\gamma_s, \quad V_z = \alpha_3\gamma_z, \quad V_n = \alpha_9(e_{ss} + e_{zz}) + \alpha_4\gamma_n + \beta_5\varphi,\\
& M_{ss} = (\alpha_5 + \alpha_6 + \alpha_7)\rho_{ss} + \alpha_5\rho_{zz}, \quad M_{zz} = \alpha_5\rho_{ss} + (\alpha_5 + \alpha_6 + \alpha_7)\rho_{zz},\\
& M_{sz} = \alpha_6\rho_{zs} + \alpha_7\rho_{sz}, \quad M_{zs} = \alpha_6\rho_{sz} + \alpha_7\rho_{zs},\\
& M_{sn} = \alpha_8\rho_{ns} + \beta_2\frac{\partial\varphi}{\partial s}, \quad M_{zn} = \alpha_8\rho_{nz} + \beta_2\frac{\partial\varphi}{\partial z},\\
& h_s = \beta_2\rho_{ns} + \beta_1\frac{\partial\varphi}{\partial s}, \quad h_z = \beta_2\rho_{nz} + \beta_1\frac{\partial\varphi}{\partial z}, \quad g = \beta_4(e_{ss} + e_{zz}) + \beta_5\gamma_n + \beta_3\varphi.
\end{aligned} \tag{2.16}
$$

The equilibrium equations (2.7)–(2.9) for the case when the assigned body loads f^i, l^i and p are null can be written in the form

$$
\begin{aligned}
& \frac{\partial}{\partial s}N_{ss} + \frac{\partial}{\partial z}N_{zs} + \frac{1}{R(s)}V_s = 0, \quad \frac{\partial}{\partial s}N_{sz} + \frac{\partial}{\partial z}N_{zz} = 0,\\
& \frac{\partial}{\partial s}V_s + \frac{\partial}{\partial z}V_z - \frac{1}{R(s)}N_{ss} = 0, \quad \frac{\partial}{\partial s}M_{ss} + \frac{\partial}{\partial z}M_{zs} - V_s = 0,\\
& \frac{\partial}{\partial s}M_{sz} + \frac{\partial}{\partial z}M_{zz} - V_z = 0, \quad \frac{\partial}{\partial s}M_{sn} + \frac{\partial}{\partial z}M_{zn} - V_n = 0,\\
& \frac{\partial}{\partial s}h_s + \frac{\partial}{\partial z}h_z - g = 0.
\end{aligned} \tag{2.17}
$$

Relations (2.15)–(2.17) represent the field equations that govern the static deformation of cylindrical Cosserat shells and they will be employed in the subsequent sections.

The expression of the strain energy density associated to cylindrical Cosserat surfaces with voids is given by

$$
\begin{aligned}
\Psi = {} & \tfrac{1}{2}\alpha_1 e_{\alpha\alpha}e_{\beta\beta} + \alpha_2 e_{\alpha\beta}e_{\alpha\beta} + \tfrac{1}{2}\alpha_3\gamma_\alpha\gamma_\alpha + \tfrac{1}{2}\alpha_4(\gamma_3)^2 + \tfrac{1}{2}(\alpha_5\rho_{\alpha\alpha}\rho_{\beta\beta} + \alpha_6\rho_{\alpha\beta}\rho_{\alpha\beta} + \alpha_7\rho_{\alpha\beta}\rho_{\beta\alpha}) + \tfrac{1}{2}\alpha_8\rho_{3\alpha}\rho_{3\alpha}\\
& + \alpha_9 e_{\alpha\alpha}\gamma_3 + \tfrac{1}{2}\beta_1\varphi_{,\alpha}\varphi_{,\alpha} + \beta_2\rho_{3\alpha}\varphi_{,\alpha} + \tfrac{1}{2}\beta_3\varphi^2 + \beta_4 e_{\alpha\alpha}\varphi + \beta_5\gamma_3\varphi.
\end{aligned} \tag{2.18}
$$

Under the hypothesis that the strain energy density Ψ is a positive definite quadratic form of the variables $e_{\alpha\beta}$, γ_i, $\rho_{i\alpha}, \varphi, \varphi_{,\alpha}$, it follows that the constitutive coefficients satisfy the restrictions:

$$\alpha_1+\alpha_2>0,\quad \alpha_2>0,\quad \alpha_3>0,\quad \alpha_4(\alpha_1+\alpha_2)-\alpha_9^2>0,$$
$$2\alpha_5+\alpha_6+\alpha_7>0,\quad \alpha_6>0,\quad \alpha_6\pm\alpha_7>0,\quad \alpha_8>0,\quad \alpha_8\beta_1-\beta_2^2>0,$$
$$\begin{vmatrix}\alpha_1+\alpha_2 & \alpha_9 & \beta_4\\ \alpha_9 & \alpha_4 & \beta_5\\ \beta_4 & \beta_5 & \beta_3\end{vmatrix}>0.$$

3. Statement of Saint-Venant's problem for Cosserat shells

Let us consider a porous shell whose reference configuration is the cylindrical surface $\mathscr{S}$ given by (2.12) and let $\mathscr{C}_0$, $\mathscr{C}_{\bar{h}}$ be the end edge curves perpendicular to the generator.

We mention that the cylindrical surface $\mathscr{S}$ can be either open (in the case when the curves $\mathscr{C}_z$ are open) or closed (when the curves $\mathscr{C}_z$ are closed). The boundary of a closed cylindrical Cosserat shell consists of the end edge curves $\mathscr{C}_0$ and $\mathscr{C}_{\bar{h}}$. For an open cylindrical shell, we denote by L_0 and $L_{\bar{s}}$ the lateral edges parallel to the generator characterized by the equations $s=0$ and $\bar{s}$, respectively. In this case, the boundary of $\mathscr{S}$ is $\partial\mathscr{S}=\mathscr{C}_0\cup\mathscr{C}_{\bar{h}}\cup L_0\cup L_{\bar{s}}$.

For porous cylindrical shells, the Saint-Venant's problem consists in finding a solution $v=\{\boldsymbol{u},\boldsymbol{\delta},\varphi\}$ of the Eqs. (2.15)–(2.17) subject to the boundary conditions on the end edges

$$\begin{aligned}&\boldsymbol{N}=\boldsymbol{N}^{(1)},\quad \boldsymbol{M}=\boldsymbol{M}^{(1)},\quad h=0\quad \text{on } \mathscr{C}_0,\\ &\boldsymbol{N}=\boldsymbol{N}^{(2)},\quad \boldsymbol{M}=\boldsymbol{M}^{(2)},\quad h=0\quad \text{on } \mathscr{C}_{\bar{h}},\end{aligned} \tag{3.1}$$

where $\boldsymbol{N}^{(\alpha)}$ and $\boldsymbol{M}^{(\alpha)}(\alpha=1,2)$ are prescribed functions. In the case of open cylindrical surfaces, the solution v must satisfy, in addition, the following null conditions on the lateral edges:

$$\boldsymbol{N}=\boldsymbol{0},\quad \boldsymbol{M}=\boldsymbol{0},\quad h=0\quad \text{on } L_0\cup L_{\bar{s}}. \tag{3.2}$$

The purpose of this paper is to determine a solution of the *relaxed* Saint-Venant's problem for cylindrical shells. In the relaxed formulation of Saint-Venant's problem the conditions $(3.1)_{1,2}$ are replaced by the following requirements:

$$\int_{\mathscr{C}_0}\boldsymbol{N}\,\mathrm{d}l=\mathscr{R}^0,\quad \int_{\mathscr{C}_0}(\boldsymbol{R}\times\boldsymbol{N}+\boldsymbol{D}\times\boldsymbol{M})\,\mathrm{d}l=\mathscr{M}^0. \tag{3.3}$$

The above relations express the conditions that the resultant of the contact forces acting on $\mathscr{C}_0$ is equal to a prescribed vector $\mathscr{R}^0$ and the resultant moment about O of the contact forces and contact director couples acting on $\mathscr{C}_0$ has the prescribed value $\mathscr{M}^0$.

As a consequence of the equilibrium equations $(2.17)_{1-6}$ and of the conditions (3.3), we deduce that

$$\int_{\mathscr{C}_{\bar{h}}}\boldsymbol{N}\,\mathrm{d}l=-\mathscr{R}^0,\quad \int_{\mathscr{C}_{\bar{h}}}(\boldsymbol{R}\times\boldsymbol{N}+\boldsymbol{D}\times\boldsymbol{M})\,\mathrm{d}l=-\mathscr{M}^0. \tag{3.4}$$

Also, in the relaxed Saint-Venant's problem the conditions $(3.1)_{3,6}$ are replaced by the requirements that the resultant equilibrated stress acting on each end edge curve is zero, i.e.

$$\int_{\mathscr{C}_0}h\,\mathrm{d}l=0,\quad \int_{\mathscr{C}_{\bar{h}}}h\,\mathrm{d}l=0. \tag{3.5}$$

On the edge curve $\mathscr{C}_0$ we have $\boldsymbol{v}=-\boldsymbol{e}_3$ and

$$\boldsymbol{N}=-(N_{zs}\boldsymbol{A}_1+N_{zz}\boldsymbol{A}_2+V_z\boldsymbol{A}_3),\quad \boldsymbol{M}=-(M_{zs}\boldsymbol{A}_1+M_{zz}\boldsymbol{A}_2+M_{zn}\boldsymbol{A}_3)\quad \text{on } \mathscr{C}_0.$$

Hence, the conditions (3.3) can be written in the form

$$\begin{aligned}&\int_{\mathscr{C}_0}(x'_\alpha N_{zs}+\epsilon_{\alpha\beta}x'_\beta V_z)\,\mathrm{d}l=-\mathscr{R}^0_\alpha,\quad \int_{\mathscr{C}_0}N_{zz}\,\mathrm{d}l=-\mathscr{R}^0_3,\\&\int_{\mathscr{C}_0}(x'_\alpha M_{zz}+\epsilon_{\beta\alpha}x_\beta N_{zz})\,\mathrm{d}l=\mathscr{M}^0_\alpha,\\&\int_{\mathscr{C}_0}(\epsilon_{\alpha\beta}x'_\alpha x_\beta N_{zs}+x_\alpha x'_\alpha V_z-M_{zs})\,\mathrm{d}l=\mathscr{M}^0_3,\end{aligned}\tag{3.6}$$

where we have denoted by $\mathscr{R}^0_i=\boldsymbol{\mathscr{R}}^0\cdot\boldsymbol{e}_i$, $\mathscr{M}^0_i=\boldsymbol{\mathscr{M}}^0\cdot\boldsymbol{e}_i$. Similarly, from the conditions (3.5) we obtain that

$$\int_{\mathscr{C}_0}h_z\,\mathrm{d}l=0,\quad \int_{\mathscr{C}_{\bar{h}}}h_z\,\mathrm{d}l=0.\tag{3.7}$$

Since L_0 and $L_{\bar{s}}$ are parallel to $\boldsymbol{e}_3$, the conditions (3.2) on the lateral edges are equivalent to

$$N_{ss}=N_{sz}=V_s=0,\quad M_{ss}=M_{sz}=M_{sn}=0,\quad h_s=0\quad \text{on } L_0\cup L_{\bar{s}}.\tag{3.8}$$

To summarize, the relaxed Saint-Venant's problem consists in the determination of a solution $v=\{\boldsymbol{u},\boldsymbol{\delta},\varphi\}$ of class $C^2(\mathscr{S})\cap C^1(\overline{\mathscr{S}})$ for the Eqs. (2.15)–(2.17) which satisfies the relations (3.6) and (3.7) and the boundary conditions on the lateral edges (3.8) (in the case of an open surface).

In the same way as in the classical theory of elasticity (see Ieşan, 1987), we remark that the relaxed Saint-Venant's problem for cylindrical shells decomposes into two problems (P_1) and (P_2) characterized by the following assumptions concerning the resultants $\boldsymbol{\mathscr{R}}^0$ and $\boldsymbol{\mathscr{M}}^0$:

(P_1): $\mathscr{R}^0_\alpha=0$,

(P_2): $\mathscr{R}^0_3=\mathscr{M}^0_i=0$.

The problem (P_1) is the extension, bending and torsion problem for cylindrical shells, while (P_2) is the flexure problem. We denote by $K_I(\mathscr{R}^0_3,\mathscr{M}^0_1,\mathscr{M}^0_2,\mathscr{M}^0_3)$ and $K_{II}(\mathscr{R}^0_1,\mathscr{R}^0_2)$ the sets of solutions of the problems (P_1) and (P_2), respectively, and by $\mathscr{D}$ the set of all elements $v=\{u,\boldsymbol{\delta},\varphi\}$ that satisfy Eqs. (2.15)–(2.17) and the boundary conditions on the lateral edges (3.8) (in the case of an open surface). For any $v=\{\boldsymbol{u},\boldsymbol{\delta},\varphi\}$, we introduce the vector-valued linear functionals $\boldsymbol{\mathscr{R}}(\cdot)$ and $\boldsymbol{\mathscr{M}}(\cdot)$ defined by

$$\boldsymbol{\mathscr{R}}(v)=\int_{\mathscr{C}_0}\boldsymbol{N}(v)\,\mathrm{d}l,\quad \boldsymbol{\mathscr{M}}(v)=\int_{\mathscr{C}_0}[\boldsymbol{R}\times\boldsymbol{N}(v)+\boldsymbol{D}\times\boldsymbol{M}(v)]\,\mathrm{d}l$$

and we designate by $\mathscr{R}_\alpha(v)=\boldsymbol{\mathscr{R}}(v)\cdot\boldsymbol{e}_\alpha$ and $\mathscr{M}_\alpha(v)=\boldsymbol{\mathscr{M}}(v)\cdot\boldsymbol{e}_\alpha$.

In what follows, we use the method established by Ieşan (1986, 1987) in order to determine a solution of the relaxed Saint-Venant's problem for cylindrical Cosserat shells. We begin with the following theorem.

Theorem 1. If $v\in\mathscr{D}$ and $\frac{\partial v}{\partial x_3}\in C^1(\bar{\mathscr{S}})$, then $\frac{\partial v}{\partial x_3}\in\mathscr{D}$ and the following relations hold:

$$\boldsymbol{\mathscr{R}}\left(\frac{\partial v}{\partial x_3}\right)=\boldsymbol{0},\quad \boldsymbol{\mathscr{M}}\left(\frac{\partial v}{\partial x_3}\right)=\epsilon_{\alpha\beta}\mathscr{R}_\beta(v)\boldsymbol{e}_\alpha.$$

Proof. In view of (2.15)–(2.17) and (3.8), it follows that $\frac{\partial v}{\partial x_3}\in\mathscr{D}$. We obtain

$$\begin{aligned}\boldsymbol{\mathscr{R}}\left(\frac{\partial v}{\partial x_3}\right)&=\int_{\mathscr{C}_0}\frac{\partial}{\partial z}\boldsymbol{N}(v)\,\mathrm{d}l=\int_{\mathscr{C}_0}-\frac{\partial}{\partial z}(N_{zs}\boldsymbol{A}_1+N_{zz}\boldsymbol{A}_2+V_z\boldsymbol{A}_3)\,\mathrm{d}l\\&=\int_{\mathscr{C}_0}\left[\left(\frac{\partial}{\partial s}N_{ss}+\sigma'V_s\right)\boldsymbol{\tau}+\frac{\partial}{\partial s}N_{sz}\boldsymbol{e}_3+\left(\frac{\partial}{\partial s}V_s-\sigma'N_{ss}\right)\boldsymbol{n}\right]\mathrm{d}l\\&=\int_{\mathscr{C}_0}\frac{\partial}{\partial s}(N_{ss}\boldsymbol{\tau}+N_{sz}\boldsymbol{e}_3+V_s\boldsymbol{n})\,\mathrm{d}l=\boldsymbol{0}.\end{aligned}$$

On the other hand, by virtue of the Eqs. (2.14) and (2.17) and of the relation $N_{zs} = N_{sz} + \sigma' M_{sz}$, we can write

$$\begin{aligned}\mathscr{M}\left(\frac{\partial v}{\partial x_3}\right) &= -\int_{\mathscr{C}_0}\left[\boldsymbol{R}\times\frac{\partial}{\partial z}(N_{zs}\boldsymbol{A}_1 + N_{zz}\boldsymbol{A}_2 + V_z\boldsymbol{A}_3) + \boldsymbol{A}_3\times\frac{\partial}{\partial z}(M_{zs}\boldsymbol{A}_1 + M_{zz}\boldsymbol{A}_2 + M_{zn}\boldsymbol{A}_3)\right]\mathrm{d}l\\ &= \int_{\mathscr{C}_0}\left\{(x_\alpha\boldsymbol{e}_\alpha)\times\left[\left(\frac{\partial}{\partial s}N_{ss} + \sigma' V_s\right)\boldsymbol{\tau} + \left(\frac{\partial}{\partial s}N_{sz}\right)\boldsymbol{e}_3 + \left(\frac{\partial}{\partial s}V_s - \sigma' N_{ss}\right)\boldsymbol{n}\right]\right.\\ &\quad\left. + \left(\frac{\partial}{\partial s}M_{ss} - V_s\right)\boldsymbol{e}_3 - \left(\frac{\partial}{\partial s}M_{sz} - V_z\right)\boldsymbol{\tau}\right\}\mathrm{d}l\\ &= \int_{\mathscr{C}_0}\frac{\partial}{\partial s}\left[(N_{ss}\epsilon_{\alpha\beta}x_\alpha x'_\beta - V_s x_\alpha x'_\alpha + M_{ss})\boldsymbol{e}_3 + (N_{sz}\epsilon_{\alpha\beta}x_\beta - M_{sz}x'_\alpha)\boldsymbol{e}_\alpha\right]\mathrm{d}l\\ &\quad - \epsilon_{\alpha\beta}\left[\int_{\mathscr{C}_0}(x'_\beta N_{zs} + \epsilon_{\beta\gamma}x'_\gamma V_z)\,\mathrm{d}l\right]\boldsymbol{e}_\alpha = \epsilon_{\alpha\beta}\mathscr{R}_\beta(v)\boldsymbol{e}_\alpha.\end{aligned}$$

This completes the proof. □

We deduce the following consequences, which will be used in the subsequent sections.

Corollary 2. If $v \in K_I(\mathscr{R}_3^0, \mathscr{M}_1^0, \mathscr{M}_2^0, \mathscr{M}_3^0)$ *and* $\frac{\partial v}{\partial x_3} \in C^1(\bar{\mathscr{S}})$, *then* $\frac{\partial v}{\partial x_3} \in \mathscr{D}$ *and*

$$\mathscr{R}\left(\frac{\partial v}{\partial x_3}\right) = \boldsymbol{0}, \quad \mathscr{M}\left(\frac{\partial v}{\partial x_3}\right) = \boldsymbol{0}.$$

Corollary 3. *If* $v \in K_{II}(\mathscr{R}_1^0, \mathscr{R}_2^0)$ *and* $\frac{\partial v}{\partial x_3} \in C^1(\overline{\mathscr{S}})$, *then*

$$\frac{\partial v}{\partial x_3} \in K_I(0, \mathscr{R}_2^0, -\mathscr{R}_1^0, 0).$$

4. Deformation of open cylindrical shells

In this section, we shall determine a solution of the relaxed Saint-Venant's problem for open cylindrical shells. The solution that we are looking for must satisfy the boundary conditions on the lateral edges (3.8), since the curve section $\mathscr{C}_z$ is open.

Without loss of generality, we can choose the origin O of the Cartesian coordinate frame such that

$$\int_0^{\bar{s}} x_\alpha(s)\,\mathrm{d}s = 0 \quad (\alpha = 1, 2). \tag{4.1}$$

4.1. Extension, bending and torsion

We now confine our attention to the problem (P_1), characterized by $\mathscr{R}_\alpha^0 = 0$.

Suggested by Corollary 2, we search for a solution $v \in K_I(\mathscr{R}_3^0, \mathscr{M}_1^0, \mathscr{M}_2^0, \mathscr{M}_3^0)$, $v = \{\boldsymbol{u}, \boldsymbol{\delta}, \varphi\}$, such that

$$\frac{\partial \boldsymbol{u}}{\partial x_3} = \boldsymbol{u}^* + \boldsymbol{\omega}\times\boldsymbol{R}, \quad \frac{\partial \boldsymbol{\delta}}{\partial x_3} = \boldsymbol{\omega}\times\boldsymbol{D}, \quad \frac{\partial\varphi}{\partial x_3} = 0, \tag{4.2}$$

where $\boldsymbol{u}^*$ and $\boldsymbol{\omega}$ are arbitrary constant vectors. We mention that any vector field $\{\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\delta}}\}$ of the form

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u}^* + \boldsymbol{\omega}\times\boldsymbol{R}, \quad \widetilde{\boldsymbol{\delta}} = \boldsymbol{\omega}\times\boldsymbol{D}, \tag{4.3}$$

represents a rigid body displacement field of the Cosserat shell (see Naghdi, 1972, p. 463).

Since $\boldsymbol{D} = \boldsymbol{A}_3 = \boldsymbol{n}$, from the relations (4.2) we find

$$u_{i,3} = u_i^* + \epsilon_{ijk}\omega_j x_k, \quad \delta_{\alpha,3} = \omega_3 x'_\alpha, \quad \delta_{3,3} = -\omega_\alpha x'_\alpha, \quad \varphi_{,3} = 0, \tag{4.4}$$

where we have denoted by $u_i = \boldsymbol{u}\cdot\boldsymbol{e}_i$, $\delta_i = \boldsymbol{\delta}\cdot\boldsymbol{e}_i$, $u_i^* = \boldsymbol{u}^*\cdot\boldsymbol{e}_i$, $\omega_i = \boldsymbol{\omega}\cdot\boldsymbol{e}_i$ and ϵ_{ijk} is the three-dimensional alternator. By integrating the equations (4.4) with respect to x_3, we obtain

$$\begin{aligned} u_\alpha &= \frac{1}{2}\epsilon_{\alpha\beta}a_\beta x_3^2 - k_0\epsilon_{\alpha\beta}x_\beta x_3 + w_\alpha(s),\\ u_3 &= \epsilon_{\alpha\beta}a_\alpha x_\beta x_3 + a_3x_3 + w_3(s), \quad \delta_\alpha = k_0x'_\alpha x_3 + \zeta_\alpha(s),\\ \delta_3 &= -a_\alpha x'_\alpha x_3 + \zeta_3(s), \quad \varphi = \varphi(s), \end{aligned} \tag{4.5}$$

except for an additive rigid body displacement field of the form (4.3). The constants a_i and k_0 that appear in (4.5) are given by $a_\alpha = \omega_\alpha$, $a_3 = u_3^*$, $k_0 = \omega_3$ and the functions $w_i(s)$, $\zeta_i(s)$ designate arbitrary functions of class $C^2[0, \bar{s}]$. In what follows, we shall determine a_i, k_0, $w_i(s)$ and $\zeta_i(s)$ such that the basic Eqs. (2.15)–(2.17) and the boundary conditions (3.6)–(3.8) be satisfied.

Let us introduce the notations

$$\begin{aligned} &\boldsymbol{w}(s) = w_\alpha(s)\boldsymbol{e}_\alpha, \quad \boldsymbol{\zeta}(s) = \zeta_\alpha(s)\boldsymbol{e}_\alpha,\\ &w_s = \boldsymbol{w}\cdot\boldsymbol{\tau}, \quad w_n = \boldsymbol{w}\cdot\boldsymbol{n}, \quad \zeta_s = \boldsymbol{\zeta}\cdot\boldsymbol{\tau}, \quad \zeta_n = \boldsymbol{\zeta}\cdot\boldsymbol{n}. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we derive

$$\begin{aligned} u_s &= \frac{1}{2}z^2\epsilon_{\alpha\beta}a_\beta x'_\alpha + k_0z\epsilon_{\alpha\beta}x_\alpha x'_\beta + w_s,\\ u_n &= \frac{1}{2}z^2a_\alpha x'_\alpha - k_0zx_\alpha x'_\alpha + w_n, \quad u_z = z(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3) + w_3,\\ \delta_s &= k_0z + \zeta_s, \quad \delta_n = \zeta_n, \quad \delta_z = -za_\alpha x'_\alpha + \zeta_3, \quad \varphi = \varphi(s). \end{aligned} \tag{4.7}$$

In order to simplify the subsequent expressions, let us introduce the notations

$$\alpha_0 = \alpha_5 + \alpha_6 + \alpha_7, \quad \beta_0 = \alpha_5\alpha_0^{-1}.$$

Making use of the geometrical equations (2.15) and of the constitutive equations (2.16), we obtain

$$\begin{aligned} &N_{ss} = \alpha_1(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3) - \alpha_5a_\alpha x'_\alpha\sigma' + \left[\alpha_1 + 2\alpha_2 + \alpha_0(\sigma')^2\right](\boldsymbol{w}'\cdot\boldsymbol{\tau}) + \alpha_0\zeta'_s\sigma' + \alpha_9\zeta_n + \beta_4\varphi,\\ &N_{zz} = (\alpha_1 + 2\alpha_2)(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3) + \alpha_1\boldsymbol{w}'\cdot\boldsymbol{\tau} + \alpha_9\zeta_n + \beta_4\varphi,\\ &N_{sz} = \alpha_2(k_0\epsilon_{\alpha\beta}x_\alpha x'_\beta + w'_3), \quad V_s = \alpha_3(\boldsymbol{w}'\cdot\boldsymbol{n} + \zeta_s),\\ &V_z = \alpha_3(-k_0x_\alpha x'_\alpha + \zeta_3), \quad V_n = \alpha_9(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3 + \boldsymbol{w}'\cdot\boldsymbol{\tau}) + \alpha_4\zeta_n + \beta_5\varphi,\\ &M_{ss} = \alpha_0\left[\sigma'(\boldsymbol{w}'\cdot\boldsymbol{\tau}) + \zeta'_s\right] - \alpha_5a_\alpha x'_\alpha, \quad M_{zz} = \alpha_5\left[\sigma'(\boldsymbol{w}'\cdot\boldsymbol{\tau}) + \zeta'_s\right] - \alpha_0a_\alpha x'_\alpha,\\ &M_{sz} = \alpha_7k_0 + \alpha_6(k_0\epsilon_{\alpha\beta}x_\alpha x'_\beta\sigma' + \zeta'_3), \quad M_{zs} = \alpha_6k_0 + \alpha_7(k_0\epsilon_{\alpha\beta}x_\alpha x'_\beta\sigma' + \zeta'_3),\\ &M_{sn} = \alpha_8\zeta'_n + \beta_2\varphi', \quad M_{zn} = 0, \quad h_s = \beta_2\zeta'_n + \beta_1\varphi',\\ &h_z = 0, \quad g = \beta_4(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3 + \boldsymbol{w}'\cdot\boldsymbol{\tau}) + \beta_5\zeta_n + \beta_3\varphi. \end{aligned} \tag{4.8}$$

Then, the equilibrium equations (2.15)–(2.17) reduce to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}N_{ss} = -\sigma'V_s, \quad \frac{\mathrm{d}}{\mathrm{d}s}N_{sz} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}s}V_s = \sigma'N_{ss},\\ &\frac{\mathrm{d}}{\mathrm{d}s}M_{ss} = V_s, \quad \frac{\mathrm{d}}{\mathrm{d}s}M_{sz} = V_z, \quad \frac{\mathrm{d}}{\mathrm{d}s}M_{sn} = V_n, \quad \frac{\mathrm{d}}{\mathrm{d}s}h_s = g. \end{aligned} \tag{4.9}$$

From the relations $(4.9)_{1,3,4}$ we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}s}(N_{ss}\boldsymbol{\tau}+V_s\boldsymbol{n})=\mathbf{0}$$

and hence, there exist the constants B_i $(i=1,2,3)$ such that

$$N_{ss}=B_\alpha x'_\alpha,\quad V_s=\epsilon_{\alpha\beta}B_\alpha x'_\beta,\quad M_{ss}=\epsilon_{\alpha\beta}B_\alpha x_\beta+B_3. \tag{4.10}$$

By using the Eqs. $(4.8)_{1,4,7}$ and (4.10) we derive, after some calculations, that the following equality holds:

$$(\alpha_1+2\alpha_2)(\boldsymbol{w}'\cdot\boldsymbol{\tau})+\alpha_9\zeta_n+\beta_4\varphi=-\alpha_1(\epsilon_{\alpha\beta}a_\alpha x_\beta+a_3)+B_\alpha(x'_\alpha+\epsilon_{\beta\alpha}x_\beta\sigma')-B_3\sigma'. \tag{4.11}$$

In view of (4.8), we see that the equations $(4.9)_{2,5}$ are equivalent to

$$\alpha_2\frac{\mathrm{d}}{\mathrm{d}s}(k_0\epsilon_{\alpha\beta}x_\alpha x'_\beta+w'_3)=0,\quad\left(\alpha_6\frac{\mathrm{d}^2}{\mathrm{d}s^2}-\alpha_3\right)(\zeta_3-k_0x_\alpha x'_\alpha)=0. \tag{4.12}$$

Also, the equilibrium equations $(4.9)_6$ and $(4.9)_7$ can be put, respectively, in the form

$$\begin{aligned}&\alpha_8\zeta''_n+\beta_2\varphi''-\alpha_4\zeta_n-\beta_5\varphi-\alpha_9(\boldsymbol{w}'\cdot\boldsymbol{\tau})=\alpha_9(\epsilon_{\alpha\beta}a_\alpha x_\beta+a_3),\\&\beta_2\zeta''_n+\beta_1\varphi''-\beta_5\zeta_n-\beta_3\varphi-\beta_4(\boldsymbol{w}'\cdot\boldsymbol{\tau})=\beta_4(\epsilon_{\alpha\beta}a_\alpha x_\beta+a_3).\end{aligned} \tag{4.13}$$

Let us impose that the boundary conditions on the lateral edges (3.8) be satisfied. Since $N_{ss}=V_s=M_{ss}=0$ for $s=0,\bar{s}$, from (4.10) it follows that:

$$B_i=0,\quad i=1,2,3.$$

On the other hand, taking into account the relations $N_{sz}=M_{sz}=0$ for $s=0,\bar{s}$ and (4.12), we obtain

$$w_3(s)=-k_0\int_0^s\epsilon_{\alpha\beta}x_\alpha x'_\beta\,\mathrm{d}s,\quad\zeta_3(s)=k_0x_\alpha x'_\alpha-k_0p_1(s), \tag{4.14}$$

where $p_1(s)$ denotes the function

$$p_1(s)=\frac{\alpha_6+\alpha_7}{\sqrt{\alpha_3\alpha_6}}\left[\cosh\left(\sqrt{\frac{\alpha_3}{\alpha_6}}\cdot\frac{\bar{s}}{2}\right)\right]^{-1}\sinh\left[\sqrt{\frac{\alpha_3}{\alpha_6}}\left(s-\frac{\bar{s}}{2}\right)\right]. \tag{4.15}$$

The conditions $M_{sn}=h_s=0$ for $s=0,\bar{s}$ are satisfied provided

$$\zeta'_n(0)=\zeta'_n(\bar{s})=0,\quad\varphi'(0)=\varphi'(\bar{s})=0. \tag{4.16}$$

We observe that we can determine the functions $(\boldsymbol{w}'\cdot\boldsymbol{\tau})(s)$, $\zeta_n(s)$ and $\varphi(s)$ from the system of Eqs. (4.11) and (4.13), together with the conditions (4.16). Indeed, let $y_{(\gamma)}(s)$, $z_{(\gamma)}(s)$ and $\varphi_{(\gamma)}(s)(\gamma=1,2)$ be the functions defined on $[0,\bar{s}]$ that satisfy the equations

$$\begin{aligned}&(\alpha_1+2\alpha_2)y_{(\gamma)}+\alpha_9z_{(\gamma)}+\beta_4\varphi_{(\gamma)}=-\alpha_1x_\gamma,\\&\alpha_8z''_{(\gamma)}+\beta_2\varphi''_{(\gamma)}-\alpha_4z_{(\gamma)}-\beta_5\varphi_{(\gamma)}-\alpha_9y_{(\gamma)}=\alpha_9x_\gamma,\\&\beta_2z''_{(\gamma)}+\beta_1\varphi''_{(\gamma)}-\beta_5z_{(\gamma)}-\beta_3\varphi_{(\gamma)}-\beta_4y_{(\gamma)}=\beta_4x_\gamma,\end{aligned} \tag{4.17}$$

for $s\in[0,\bar{s}],\gamma=1,2$, and are subject to the conditions

$$z'_{(\gamma)}(0)=z'_{(\gamma)}(\bar{s})=0,\quad\varphi'_{(\gamma)}(0)=\varphi'_{(\gamma)}(\bar{s})=0,\quad\gamma=1,2. \tag{4.18}$$

We mention that the solution $\{y_{(\gamma)},z_{(\gamma)},\varphi_{(\gamma)}\}$ of the boundary value problem (4.17) and (4.18) is uniquely determined and it can be computed with the help of the variation of constants method (see e.g., Reid, 1971).

We denote by A, B and Γ the constants that verify the system of equations

$$\begin{aligned}&(\alpha_1 + 2\alpha_2)A + \alpha_9 B + \beta_4 \Gamma = -\alpha_1,\\ &\alpha_9 A + \alpha_4 B + \beta_5 \Gamma = -\alpha_9,\\ &\beta_4 A + \beta_5 B + \beta_3 \Gamma = -\beta_4.\end{aligned} \tag{4.19}$$

With these notations, the functions $(\boldsymbol{w}' \cdot \boldsymbol{\tau})$, ζ_n and φwhich satisfy the Eqs. (4.11) and (4.13) (written with $B_i = 0$) and the conditions (4.16) can be expressed in the form

$$\begin{aligned}&\boldsymbol{w}' \cdot \boldsymbol{\tau} = a_3 A + \epsilon_{\alpha\beta} a_\alpha y_{(\beta)},\\ &\zeta_n = a_3 B + \epsilon_{\alpha\beta} a_\alpha z_{(\beta)},\\ &\varphi = a_3 \Gamma + \epsilon_{\alpha\beta} a_\alpha \varphi_{(\beta)}.\end{aligned} \tag{4.20}$$

In the same time, from the Eqs. $(4.8)_{4,7}$ and $(4.10)_{2,3}$ we deduce that

$$\zeta_s = -\boldsymbol{w}' \cdot \boldsymbol{n}, \quad \boldsymbol{w}'' \cdot \boldsymbol{n} = -\beta_0 a_\alpha x'_\alpha. \tag{4.21}$$

With the help of the relations $(4.20)_{1,2}$ and (4.21) we determine the functions $w_\alpha(s)$ and $\zeta_\alpha(s)$. The result is

$$w_\alpha(s) = W_\alpha[a_i](s), \quad \zeta_\alpha(s) = Z_\alpha[a_i](s),$$

where the functions W_α, Z_α depend on the constants a_i $(i = 1, 2, 3)$ and are defined by

$$\begin{aligned}W_\alpha[a_i](s) &= a_3 A\left(x_\alpha + \epsilon_{\alpha\beta}\int_0^s x'_\beta \sigma\,\mathrm{d}s\right) - a_\gamma\left[\beta_0 \epsilon_{\alpha\beta}\int_0^s x'_\beta x_\gamma\,\mathrm{d}s + \epsilon_{\beta\gamma}\int_0^s x'_\alpha y_{(\beta)}\,\mathrm{d}s\right.\\ &\qquad\left.-\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\left(x_\beta\int_0^s y_{(\delta)}\sigma'\,\mathrm{d}s - \int_0^s x_\beta y_{(\delta)}\sigma'\,\mathrm{d}s\right)\right],\\ Z_\alpha[a_i](s) &= a_3\left(-A x'_\alpha \sigma + B\epsilon_{\alpha\beta}x'_\beta\right) + a_\gamma\left(\beta_0 x'_\alpha x_\gamma + \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}x'_\beta z_{(\delta)} + \epsilon_{\beta\gamma}x'_\alpha\int_0^s y_{(\beta)}\sigma'\,\mathrm{d}s\right).\end{aligned} \tag{4.22}$$

In what follows, we shall find the values of the constants a_i and k_0 such that the boundary conditions on the end edges (3.6) and (3.7) be satisfied. By virtue of the relations $N_{zs} = N_{sz} + \sigma' M_{sz}$,$N_{sz} = 0$ and $(4.9)_5$, we get

$$\int_{\mathscr{C}_0}(x'_\alpha N_{zs} + \epsilon_{\alpha\beta}x'_\beta V_z)\,\mathrm{d}l = \int_0^{\bar{s}}(x'_\alpha \sigma' M_{sz} + \epsilon_{\alpha\beta}x'_\beta V_z)\,\mathrm{d}s = \int_0^{\bar{s}}\frac{\mathrm{d}}{\mathrm{d}s}(\epsilon_{\alpha\beta}x'_\beta M_{sz})\,\mathrm{d}s = 0$$

and then the conditions $(3.6)_1$ (written for $\mathscr{R}^0_\alpha = 0$) hold. The conditions (3.7) are also satisfied, since $h_z = 0$. By integrating the equations (4.17) from $s = 0$ to $\bar{s}$ we obtain

$$\int_0^{\bar{s}} y_{(\gamma)}\,\mathrm{d}s = \int_0^{\bar{s}} z_{(\gamma)}\,\mathrm{d}s = \int_0^{\bar{s}} \varphi_{(\gamma)}\,\mathrm{d}s = 0, \quad \gamma = 1, 2. \tag{4.23}$$

In view of (4.11), the condition $(3.6)_2$ can be written in the form

$$2\alpha_2\int_0^{\bar{s}}(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3 - \boldsymbol{w}' \cdot \boldsymbol{\tau})\,\mathrm{d}s = -\mathscr{R}^0_3.$$

By substituting in the above relation $(4.20)_1$, we find

$$a_3 = \frac{\mathscr{R}^0_3}{2\alpha_2(A-1)\bar{s}}. \tag{4.24}$$

Since we have

$$N_{zz} = 2\alpha_2(\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3 - \boldsymbol{w}' \cdot \boldsymbol{\tau}), \quad M_{zz} = (\alpha_5\beta_0 - \alpha_0)a_\alpha x'_\alpha,$$

the boundary conditions $(3.6)_3$ are equivalent to

$$I_{\alpha\beta}a_\beta = \mathscr{M}^0_\alpha, \tag{4.25}$$

where we have denoted by $I_{\alpha\beta}$ the expressions

$$I_{\alpha\beta} = (\alpha_5\beta_0 - \alpha_0)\int_0^{\bar{s}} x'_\alpha x'_\beta\,\mathrm{d}s - 2\alpha_2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\int_0^{\bar{s}} x_\gamma(x_\delta - y_{(\delta)})\,\mathrm{d}s. \tag{4.26}$$

Finally, the condition $(3.6)_4$ reduces to

$$\int_0^{\bar{s}} (M_{sz} + M_{zs})\,\mathrm{d}s = -\mathscr{M}^0_3\ .$$

In view of $(4.14)_2$ and (4.15), this relation becomes

$$k_0 = -\frac{\mathscr{M}^0_3}{2(\alpha_6+\alpha_7)}\left[\bar{s} - \frac{\alpha_6+\alpha_7}{\sqrt{\alpha_3\alpha_6}}\tanh\left(\sqrt{\frac{\alpha_3}{\alpha_6}}\cdot\frac{\bar{s}}{2}\right)\right]^{-1}. \tag{4.27}$$

Thus, we have obtained the following solution of the extension, bending and torsion problem for open cylindrical shells

$$\begin{aligned}
u_\alpha &= \frac{1}{2}\epsilon_{\alpha\beta}a_\beta x_3^2 - k_0\epsilon_{\alpha\beta}x_\beta x_3 + W_\alpha[a_i](s),\\
u_3 &= \epsilon_{\alpha\beta}a_\alpha x_\beta x_3 + a_3x_3 - k_0\int_0^s \epsilon_{\alpha\beta}x_\alpha x'_\beta\,\mathrm{d}s,\\
\delta_\alpha &= k_0x'_\alpha x_3 + Z_\alpha[a_i](s),\\
\delta_3 &= -a_\alpha x'_\alpha x_3 + k_0x_\alpha x'_\alpha - k_0p_1(s),\\
\varphi &= a_3\Gamma + \epsilon_{\alpha\beta}a_\alpha\varphi_{(\beta)}(s).
\end{aligned} \tag{4.28}$$

The constants $a_i (i = 1,2,3)$ and k_0 that appear in the above relations are determined by (4.24)–(4.27). We remind that the functions W_α, Z_α and p_1 are defined by (4.22) and (4.15), while $\varphi_{(\beta)}$ can be determined by solving the problem (4.17) and (4.18).

The solution (4.28) of the problem (P_1) will be denoted by $v(a_i, k_0)$.

4.2. Flexure

We consider now the problem (P_2), i.e. the case when $\mathscr{R}^0_3 = \mathscr{M}^0_i = 0$.

In view of Corollary 3, it is natural to search for a solution $v^0 = \{\boldsymbol{u}^0, \boldsymbol{\delta}^0, \varphi^0\}$ of the flexure problem in the form

$$v^0 = \int_0^{x_3} v(b_i, k_1)\,\mathrm{d}x_3 + v(b^0_i, k_2) + w^0, \tag{4.29}$$

where b_i, b^0_i $(i = 1,2,3)$ and k_α $(\alpha = 1,2)$ are constants, while $w^0(s) = \{\boldsymbol{f}_0(s), \boldsymbol{g}_0(s), \psi_0(s)\}$ is a function of class $C^2[0,\bar{s}]$ which depends only on s. In what follows, we shall determine b_i, b^0_i, k_α and the unknown functions $\boldsymbol{f}_0(s)$, $\boldsymbol{g}_0(s)$, $\psi_0(s)$ such that $v^0 \in K_{\mathrm{II}}(\mathscr{R}^0_1, \mathscr{R}^0_2)$.

Taking into account (4.2) and (4.29), we remark that $\frac{\partial v^0}{\partial x_3}$ differs from $v(b_i, k_1)$ only by a vector field of the form $\{\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\delta}}, 0\}$, where $\{\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{\delta}}\}$ represents a rigid body displacement field of the Cosserat surface (see (4.3)). Then, by virtue of Corollary 3, we have

$$v(b_i, k_1) \in K_{\mathrm{I}}(0, \mathscr{R}^0_2, -\mathscr{R}^0_1, 0). \tag{4.30}$$

Making use of the relations (4.24)–(4.27) written for the solution of the problem (P_1) specified in (4.30), we obtain

$$I_{\alpha\beta}b_\beta = \epsilon_{\alpha\beta}\mathscr{R}^0_\beta, \quad b_3 = 0, \quad k_1 = 0. \tag{4.31}$$

With the help of (4.28), the equality (4.29) can be written in the form

$$\begin{aligned}
u^0_\alpha &= \frac{1}{6}\epsilon_{\alpha\beta}b_\beta x_3^3 + \frac{1}{2}\epsilon_{\alpha\beta}b^0_\beta x_3^2 - k_2\epsilon_{\alpha\beta}x_\beta x_3 + x_3 W_\alpha[b_i](s) + W_\alpha[b^0_i](s) + f^0_\alpha(s),\\
u^0_3 &= \frac{1}{2}\epsilon_{\alpha\beta}b_\alpha x_\beta x_3^2 + \epsilon_{\alpha\beta}b^0_\alpha x_\beta x_3 + b^0_3 x_3 - k_2\int_0^s \epsilon_{\alpha\beta}x_\alpha x'_\beta\,\mathrm{d}s + f^0_3(s),\\
\delta^0_\alpha &= k_2 x'_\alpha x_3 + x_3 Z_\alpha[b_i](s) + Z_\alpha[b^0_i](s) + g^0_\alpha(s),\\
\delta^0_3 &= -\frac{1}{2}b_\alpha x'_\alpha x_3^2 - b^0_\alpha x'_\alpha x_3 + k_2 x_\alpha x'_\alpha - k_2 p_1(s) + g^0_3(s),\\
\varphi^0 &= \epsilon_{\alpha\beta}b_\alpha\varphi_{(\beta)}(s)x_3 + \epsilon_{\alpha\beta}b^0_\alpha\varphi_{(\beta)}(s) + b^0_3\Gamma + \psi_0(s),
\end{aligned} \tag{4.32}$$

where $f^0_i(s) = \boldsymbol{f}_0(s)\cdot\boldsymbol{e}_i$ and $g^0_i(s) = \boldsymbol{g}_0(s)\cdot\boldsymbol{e}_i$.

By virtue of the relations (4.32), (2.15) and (2.16), the equilibrium equations $(2.17)_{1,3,4,6,7}$ can be put in the equivalent form

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\left\{(\alpha_1+2\alpha_2)(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) + \alpha_9(\boldsymbol{g}_0\cdot\boldsymbol{n}) + \alpha_0\sigma'\left[\sigma'(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) + (\boldsymbol{g}_0\cdot\boldsymbol{\tau})'\right] + \beta_4\psi_0\right\} + \alpha_3\sigma'(\boldsymbol{f}'_0\cdot\boldsymbol{n} + \boldsymbol{g}_0\cdot\boldsymbol{\tau}) = 0,\\
&\alpha_3\frac{\mathrm{d}}{\mathrm{d}s}(\boldsymbol{f}'_0\cdot\boldsymbol{n} + \boldsymbol{g}_0\cdot\boldsymbol{\tau}) - \sigma'\left\{(\alpha_1+2\alpha_2)(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) + \alpha_9(\boldsymbol{g}_0\cdot\boldsymbol{n}) + \alpha_0\sigma'\left[\sigma'(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) + (\boldsymbol{g}_0\cdot\boldsymbol{\tau})'\right] + \beta_4\psi_0\right\} = 0,\\
&\alpha_0\frac{\mathrm{d}}{\mathrm{d}s}\left[\sigma'(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) + (\boldsymbol{g}_0\cdot\boldsymbol{\tau})'\right] - \alpha_3(\boldsymbol{f}'_0\cdot\boldsymbol{n} + \boldsymbol{g}_0\cdot\boldsymbol{\tau}) = 0,\\
&\alpha_8(\boldsymbol{g}_0\cdot\boldsymbol{n})'' + \beta_2\psi''_0 - \alpha_4(\boldsymbol{g}_0\cdot\boldsymbol{n}) - \beta_5\psi_0 - \alpha_9(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) = 0,\\
&\beta_2(\boldsymbol{g}_0\cdot\boldsymbol{n})'' + \beta_1\psi''_0 - \beta_5(\boldsymbol{g}_0\cdot\boldsymbol{n}) - \beta_3\psi_0 - \beta_4(\boldsymbol{f}'_0\cdot\boldsymbol{\tau}) = 0.
\end{aligned} \tag{4.33}$$

Following a procedure already employed in Section 4.1, from the system of equations (4.33) together with the conditions on the lateral edges:

$$N_{ss} = V_s = M_{ss} = M_{sn} = h_s = 0 \quad \text{for } s = 0, \bar{s},$$

we derive that

$$f^0_\alpha(s) = g^0_\alpha(s) = \psi_0(s) = 0, \tag{4.34}$$

where we have neglected a rigid body displacement field of the Cosserat shell.

The equilibrium equation $(2.17)_2$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}(f^{0\prime}_3(s) + x'_\alpha W_\alpha[b_i](s)) + 2\epsilon_{\alpha\beta}b_\alpha(x_\beta - y_{(\beta)}) = 0.$$

Taking into account the condition $N_{sz} = 0$ for $s = 0$, $\bar{s}$, from the above equation we find

$$f^0_3(s) = -\int_0^s x'_\alpha W_\alpha[b_i](s)\,\mathrm{d}s - 2\epsilon_{\alpha\beta}b_\alpha\int_0^s\int_0^\xi [x_\beta(\lambda) - y_{(\beta)}(\lambda)]\,\mathrm{d}\lambda\,\mathrm{d}\xi. \tag{4.35}$$

In order to determine $g^0_3(s)$, let us introduce the function $p_0(s)$ defined by the equality

$$g^0_3(s) = p_0(s) - \epsilon_{\alpha\beta}x'_\beta W_\alpha[b_i](s). \tag{4.36}$$

In view of the geometrical and constitutive relations (2.15) and (2.17), we deduce that the equation of equilibrium $(2.17)_5$ reduces to

$$\alpha_6 p_0''(s) - \alpha_3 p_0(s) = (\alpha_6 + \alpha_7) b_\alpha (x'_\alpha + \epsilon_{\alpha\beta} y_{(\beta)} \sigma'), \quad s \in [0, \bar{s}]. \tag{4.37}$$

The conditions on the lateral edges $M_{sz} = 0$ are equivalent to the relations

$$p_0'(0) = -\frac{(\alpha_6 + \alpha_7)}{\alpha_6} b_\alpha \beta_0 x_\alpha(0), p_0'(\bar{s}) = -\frac{(\alpha_6 + \alpha_7)}{\alpha_6} b_\alpha \left[\beta_0 x_\alpha(\bar{s}) - \epsilon_{\alpha\beta} \int_0^{\bar{s}} y_{(\beta)} \sigma' \, ds \right]. \tag{4.38}$$

By solving the boundary value problem (4.37) and (4.38) we find the function $p_0(s)$.

The solution v^0 must also verify the end edge conditions (3.6) and (3.7). We remark that $(3.6)_1$ and (3.7) are satisfied, by virtue of the relations $(4.31)_1$ and (4.23), respectively. On the other hand, from the conditions $(3.6)_{2,3}$ written for $\mathscr{R}_3^0 = \mathscr{M}_\alpha^0 = 0$ we obtain $b_i^0 = 0$ $(i = 1, 2, 3)$. Finally, the end edge condition $(3.6)_4$ written for $\mathscr{M}_3^0 = 0$ yields an equation for the determination of the constant k_2. We find

$$\begin{aligned} k_2 = \Bigg\{ & 2\epsilon_{\alpha\beta} b_\alpha \int_0^{\bar{s}} \int_0^s y_{(\beta)}(\xi) \sigma'(\xi) \, d\xi \, ds - \beta_0 b_\gamma \int_0^{\bar{s}} x_\alpha x'_\alpha x'_\gamma \, ds \\ & + \frac{2\alpha_2}{\alpha_6 + \alpha_7} \epsilon_{\gamma\delta} b_\gamma \int_0^{\bar{s}} (x_\delta - y_{(\delta)}) \int_0^s \epsilon_{\alpha\beta} x'_\alpha(\xi) x_\beta(\xi) \, d\xi \, ds + [p_0(0) - p_0(\bar{s})] \Bigg\} \\ & \times \left[2\bar{s} - \frac{2(\alpha_6 + \alpha_7)}{\sqrt{\alpha_3 \alpha_6}} \tanh\left(\sqrt{\frac{\alpha_3}{\alpha_6}} \cdot \frac{\bar{s}}{2} \right) \right]^{-1}. \end{aligned} \tag{4.39}$$

We conclude that the solution v^0 of the flexure problem for open cylindrical shells is given by

$$\begin{aligned} u_\alpha^0 &= \frac{1}{6} \epsilon_{\alpha\beta} b_\beta x_3^3 - k_2 \epsilon_{\alpha\beta} x_\beta x_3 + x_3 W_\alpha[b_i](s), \\ u_3^0 &= \frac{1}{2} \epsilon_{\alpha\beta} b_\alpha x_\beta x_3^2 - k_2 \int_0^s \epsilon_{\alpha\beta} x_\alpha x'_\beta \, ds + f_3^0(s), \\ \delta_\alpha^0 &= k_2 x'_\alpha x_3 + x_3 Z_\alpha[b_i](s), \\ \delta_3^0 &= -\frac{1}{2} b_\alpha x'_\alpha x_3^2 + k_2 x_\alpha x'_\alpha - \epsilon_{\alpha\beta} x'_\beta W_\alpha[b_i](s) - k_2 p_1(s) + p_0(s), \\ \varphi^0 &= \epsilon_{\alpha\beta} b_\alpha \varphi_{(\beta)}(s) x_3. \end{aligned} \tag{4.40}$$

5. Deformation of closed cylindrical shells

In the case of closed cylindrical shells, we denote by $\mathscr{A}_c$ the area bounded by the closed curve section $\mathscr{C}_z$. The length of the perimeter of the curve $\mathscr{C}_z$ is $\bar{s}$. Without loss of generality, the origin O of the Cartesian coordinate frame is fixed such that (4.1) holds, while the arc parameter s along the closed curve $\mathscr{C}_z$ is chosen such that

$$\sigma(\bar{s}) = \sigma(0) + 2\pi. \tag{5.1}$$

Since $\boldsymbol{u}$, $\boldsymbol{\delta}$, φ and their derivatives are single-valued functions, they satisfy the conditions

$$\begin{aligned} & \boldsymbol{u}(0, z) = \boldsymbol{u}(\bar{s}, z), \quad \boldsymbol{\delta}(0, z) = \boldsymbol{\delta}(\bar{s}, z), \quad \varphi(0, z) = \varphi(\bar{s}, z), \\ & \frac{\partial^k \boldsymbol{u}}{\partial s^k}(0, z) = \frac{\partial^k \boldsymbol{u}}{\partial s^k}(\bar{s}, z), \quad \frac{\partial^k \boldsymbol{\delta}}{\partial s^k}(0, z) = \frac{\partial^k \boldsymbol{\delta}}{\partial s^k}(\bar{s}, z), \quad \frac{\partial^k \varphi}{\partial s^k}(0, z) = \frac{\partial^k \varphi}{\partial s^k}(\bar{s}, z), \end{aligned} \tag{5.2}$$

where $k = 1, 2$ and $z \in [0, \bar{h}]$.

5.1. Extension, bending and torsion

In order to investigate the problem (P_1), we employ the same method as in Section 4.1. We search for a solution $v \in K_1(\mathscr{R}_3^0, \mathscr{M}_1^0, \mathscr{M}_2^0, \mathscr{M}_3^0)$, $v = \{\boldsymbol{u}, \boldsymbol{\delta}, \varphi\}$, such that (4.2) is satisfied. Then, we deduce the relations (4.4)–(4.13), which hold true also in the case of closed cylindrical shells.

The conditions (5.2) imply that

$$w_3(0) = w_3(\bar{s}), \quad \zeta_3(0) = \zeta_3(\bar{s}), \quad \zeta_3'(0) = \zeta_3'(\bar{s}). \tag{5.3}$$

From (4.12) and (5.3) we obtain

$$w_3(s) = -k_0 \int_0^s \epsilon_{\alpha\beta} x_\alpha x_\beta' \, \mathrm{d}s + k_0 \frac{2\mathscr{A}_c}{\bar{s}} s, \quad \zeta_3(s) = k_0 x_\alpha x_\alpha', \tag{5.4}$$

where we have neglected an additive constant representing a rigid body displacement field.

We remark that the conditions (5.2) also yield

$$\zeta_n(0) = \zeta_n(\bar{s}), \quad \zeta_n'(0) = \zeta_n'(\bar{s}), \quad \varphi(0) = \varphi(\bar{s}), \quad \varphi'(0) = \varphi'(\bar{s}). \tag{5.5}$$

The system of equations (4.11) and (4.13) and the conditions (5.5) represent a boundary value problem for the determination of the functions $(\boldsymbol{w}' \cdot \boldsymbol{\tau})(s)$, $\zeta_n(s)$ and $\varphi(s)$. Indeed, let A,B and C be the constants defined by (4.19) and let us denote, in this section, by $y_{(\gamma)}(s)$, $z_{(\gamma)}(s)$ and $\varphi_{(\gamma)}(s)$ $(s \in [0, \bar{s}], \gamma = 1, 2)$ the functions which verify the equations (4.17) and the following conditions:

$$z_{(\gamma)}(0) = z_{(\gamma)}(\bar{s}), \quad z_{(\gamma)}'(0) = z_{(\gamma)}'(\bar{s}), \quad \varphi_{(\gamma)}(0) = \varphi_{(\gamma)}(\bar{s}), \quad \varphi_{(\gamma)}'(0) = \varphi_{(\gamma)}'(\bar{s}). \tag{5.6}$$

Also, let $F_{(i)}(s)$, $G_{(i)}(s)$ and $\Phi_{(i)}(s)$ $(i = 1, 2, 3)$ be the functions defined on $[0, \bar{s}]$ which satisfy the systems of equations

$$\begin{aligned}
&\alpha_8 G_{(i)}'' + \beta_2 \Phi_{(i)}'' - \alpha_4 G_{(i)} - \beta_5 \Phi_{(i)} - \alpha_9 F_{(i)} = 0, \\
&\beta_2 G_{(i)}'' + \beta_1 \Phi_{(i)}'' - \beta_5 G_{(i)} - \beta_3 \Phi_{(i)} - \beta_4 F_{(i)} = 0, \\
&(\alpha_1 + 2\alpha_2) F_{(\alpha)} + \alpha_9 G_{(\alpha)} + \beta_4 \Phi_{(\alpha)} = x_\alpha' + \epsilon_{\beta\alpha} x_\beta \sigma' \quad (\alpha = 1, 2), \\
&(\alpha_1 + 2\alpha_2) F_{(3)} + \alpha_9 G_{(3)} + \beta_4 \Phi_{(3)} = -\sigma',
\end{aligned} \tag{5.7}$$

together with the boundary conditions

$$\begin{aligned}
&G_{(i)}(0) = G_{(i)}(\bar{s}), \quad G_{(i)}'(0) = G_{(i)}'(\bar{s}), \\
&\Phi_{(i)}(0) = \Phi_{(i)}(\bar{s}), \quad \Phi_{(i)}'(0) = \Phi_{(i)}'(\bar{s}) \quad (i = 1, 2, 3).
\end{aligned} \tag{5.8}$$

We mention that the functions $y_{(\gamma)}$, $z_{(\gamma)}$, $\varphi_{(\gamma)}$, $F_{(i)}$, $G_{(i)}$ and $\Phi_{(i)}$ exist and are uniquely determined (see e.g., Reid, 1971). With the help of these notations, the solution of the problem (4.11), (4.13), (5.5) can be written as follows:

$$\begin{aligned}
&\boldsymbol{w}' \cdot \boldsymbol{\tau} = a_3 A + \epsilon_{\alpha\beta} a_\alpha y_{(\beta)} + B_i F_{(i)}, \\
&\zeta_n = a_3 B + \epsilon_{\alpha\beta} a_\alpha z_{(\beta)} + B_i G_{(i)}, \\
&\varphi = a_3 \Gamma + \epsilon_{\alpha\beta} a_\alpha \varphi_{(\beta)} + B_i \Phi_{(i)}.
\end{aligned} \tag{5.9}$$

In an analogous manner as in Section 4.1, we derive the relations

$$\begin{aligned}
&\zeta_s = -\boldsymbol{w}' \cdot \boldsymbol{n} + \alpha_3^{-1} \epsilon_{\alpha\beta} B_\alpha x_\beta', \\
&\boldsymbol{w}'' \cdot \boldsymbol{n} = -\beta_0 a_\alpha x_\alpha' - \alpha_0^{-1} (\epsilon_{\alpha\beta} B_\alpha x_\beta + B_3) + \alpha_3^{-1} B_\alpha x_\alpha' \sigma'.
\end{aligned} \tag{5.10}$$

In view of $(5.9)_{1,2}$ and (5.10), we can determine $w_\alpha(s)$ and $\zeta_\alpha(s)$. We obtain

$$w_\alpha(s) = U_\alpha[a_i, B_i](s), \quad \zeta_\alpha(s) = Y_\alpha[a_i, B_i](s), \tag{5.11}$$

where the functions U_α and Y_α depend on the constants a_i, B_i and are defined by

$$\begin{aligned}
U_\alpha[a_i,B_i](s) &= a_3 A\left(x_\alpha + \epsilon_{\alpha\beta}\int_0^s x'_\beta \sigma\,\mathrm{d}s\right)\\
&\quad - a_\gamma\left[\beta_0\epsilon_{\alpha\beta}\int_0^s x'_\beta x_\gamma\,\mathrm{d}s + \epsilon_{\beta\gamma}\int_0^s x'_\alpha y_{(\beta)}\,\mathrm{d}s - \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\left(x_\beta\int_0^s y_{(\delta)}\sigma'\,\mathrm{d}s - \int_0^s x_\beta y_{(\delta)}\sigma'\,\mathrm{d}s\right)\right]\\
&\quad + B_\gamma\alpha_0^{-1}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\left(\int_0^s x_\beta x_\delta\,\mathrm{d}s - x_\beta\int_0^s x_\delta\,\mathrm{d}s + \alpha_0\alpha_3^{-1}\int_0^s x'_\beta x'_\delta\,\mathrm{d}s\right) + B_3\alpha_0^{-1}\epsilon_{\alpha\beta}\left(\int_0^s x_\beta\,\mathrm{d}s - s x_\beta\right)\\
&\quad + B_i\left(\int_0^s x'_\alpha F_{(i)}\,\mathrm{d}s + \epsilon_{\alpha\beta}x_\beta\int_0^s F_{(i)}\sigma'\,\mathrm{d}s - \epsilon_{\alpha\beta}\int_0^s x_\beta F_{(i)}\sigma'\,\mathrm{d}s\right),\\
Y_\alpha[a_i,B_i](s) &= a_3(B\epsilon_{\alpha\beta}x'_\beta - A x'_\alpha\sigma) + a_\gamma\left(\beta_0 x'_\alpha x_\gamma + \epsilon_{\alpha\beta}\epsilon_{\gamma\delta}x'_\beta z_{(\delta)} - \epsilon_{\gamma\delta}x'_\alpha\int_0^s y_{(\delta)}\sigma'\,\mathrm{d}s\right)\\
&\quad - B_\gamma\alpha_0^{-1}\epsilon_{\beta\gamma}x'_\alpha\int_0^s x_\beta\,\mathrm{d}s + B_3\alpha_0^{-1}s x'_\alpha + B_i\left(\epsilon_{\alpha\beta}x'_\beta G_{(i)} - x'_\alpha\int_0^s F_{(i)}\sigma'\,\mathrm{d}s\right).
\end{aligned}\tag{5.12}$$

We remark that the conditions (5.2) are satisfied if and only if the following relations hold:

$$\zeta_s(0) = \zeta_s(\bar{s}),\quad w_\alpha(0) = w_\alpha(\bar{s}),\quad \alpha = 1,2. \tag{5.13}$$

Taking into account (5.11) and (5.12), the equalities (5.13) reduce to the following conditions on the constants a_i and B_i:

$$\begin{aligned}
&a_3 2\pi A + a_\alpha\epsilon_{\alpha\beta}\int_0^{\bar{s}} y_{(\beta)}\sigma'\,\mathrm{d}s + B_i\int_0^{\bar{s}} F_{(i)}\sigma'\,\mathrm{d}s - B_3\alpha_0^{-1}\bar{s} = 0,\\
&U_\alpha[a_i,B_i](0) = U_\alpha[a_i,B_i](\bar{s}).
\end{aligned}\tag{5.14}$$

The end edge conditions (3.6) and (3.7) must also be satisfied. In the same way as in the case of open cylindrical surfaces, we find that $(3.6)_1$ and (3.7) are verified.

By virtue of (4.11) and $(5.9)_1$, the equality $(3.6)_2$ can be written in the form

$$a_3[2\alpha_2(A-1)\bar{s}] + B_\alpha A\epsilon_{\beta\alpha}\int_0^{\bar{s}} x_\beta\sigma'\,\mathrm{d}s - B_3 2\pi A = \mathscr{R}_3^0. \tag{5.15}$$

By substituting the appropriate constitutive relations into $(3.6)_3$, we deduce

$$\begin{aligned}
&I_{\alpha\beta}a_\beta - B_\gamma\left[\beta_0\epsilon_{\beta\gamma}\int_0^{\bar{s}} x'_\alpha x_\beta\,\mathrm{d}s + \epsilon_{\alpha\beta}\int_0^{\bar{s}} x_\beta(x'_\gamma + \epsilon_{\delta\gamma}x_\delta\sigma')\,\mathrm{d}s\right]\\
&\quad + B_3\epsilon_{\alpha\beta}\int_0^{\bar{s}} x_\beta\sigma'\,\mathrm{d}s + B_i\left(2\alpha_2\epsilon_{\alpha\beta}\int_0^{\bar{s}} x_\beta F_{(i)}\,\mathrm{d}s\right) = \mathscr{M}_\alpha^0,
\end{aligned}\tag{5.16}$$

where $I_{\alpha\beta}$ represents the notation introduced in (4.26).

The relations (5.14)–(5.16) form a system of six algebraic equations for the determination of the constants a_i and B_i $(i = 1,2,3)$.

The value of the constant k_0 can be computed from the end edge condition $(3.6)_4$. After some calculations, we find

$$k_0 = -\mathscr{M}_3^0\left[\alpha_2\frac{4}{\bar{s}}\mathscr{A}_c^2 + 2(\alpha_6+\alpha_7)\bar{s}\right]^{-1}. \tag{5.17}$$

In conclusion, we have obtained the following solution of the extension, bending and torsion problem for closed cylindrical shells:

$$\begin{aligned}
u_\alpha &= \frac{1}{2}\epsilon_{\alpha\beta}a_\beta x_3^2 - k_0\epsilon_{\alpha\beta}x_\beta x_3 + U_\alpha[a_i,B_i](s),\\
u_3 &= \epsilon_{\alpha\beta}a_\alpha x_\beta x_3 + a_3x_3 - k_0\int_0^s \epsilon_{\alpha\beta}x_\alpha x'_\beta\,\mathrm{d}s + k_0\frac{2\mathscr{A}_c}{\bar{s}}s,\\
\delta_\alpha &= k_0x'_\alpha x_3 + Y_\alpha[a_i,B_i](s),\quad \delta_3 = -a_\alpha x'_\alpha x_3 + k_0x_\alpha x'_\alpha,\\
\varphi &= a_3\Gamma + \epsilon_{\alpha\beta}a_\alpha\varphi_{(\beta)}(s) + B_i\Phi_{(i)}(s).
\end{aligned} \tag{5.18}$$

5.2. Circular cylindrical shells

In this section, we apply the theoretical results established previously and find the solution of the relaxed Saint-Venant's problem for closed circular cylindrical Cosserat surfaces.

We denote by R_0 the radius of the cylindrical shell in the reference configuration $\mathscr{S}$. The parametric equation of $\mathscr{S}$ is given by (2.12), where

$$x_1(s) = R_0\cos\frac{s}{R_0},\quad x_2(s) = R_0\sin\frac{s}{R_0},\qquad s\in[0,2\pi R_0]. \tag{5.19}$$

Then, from the relations (2.13) and (2.14) we deduce that

$$x'_\alpha = -R_0^{-1}\epsilon_{\alpha\beta}x_\beta,\quad x''_\alpha = -R_0^{-2}x_\alpha,\quad \sigma(s) = R_0^{-1}s + \frac{\pi}{2}. \tag{5.20}$$

We consider the problem (P_1) (i.e. $\mathscr{R}_\alpha^0 = 0$) as a particular case of the problem solved in Section 5.1. Taking into account (5.18)–(5.20) the solution of the extension, bending and torsion problem for circular cylindrical shells is given by

$$\begin{aligned}
u_\alpha &= \frac{1}{2}\epsilon_{\alpha\beta}a_\beta x_3^2 - k_0\epsilon_{\alpha\beta}x_\beta x_3 - (A^*\epsilon_{\beta\gamma}a_\beta x_\gamma + a_3A_0)x_\alpha,\\
u_3 &= (\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3)x_3,\\
\delta_\alpha &= -R_0^{-1}\left[(D^*a_\gamma x_\gamma + k_0x_3)\epsilon_{\alpha\beta}x_\beta + (B^*\epsilon_{\beta\gamma}a_\beta x_\gamma + a_3B_0)x_\alpha\right],\\
\delta_3 &= R_0^{-1}\epsilon_{\alpha\beta}a_\alpha x_\beta x_3,\quad \varphi = -(C^*\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3C_0).
\end{aligned} \tag{5.21}$$

Here, the constants a_i and k_0 have the following values:

$$\begin{aligned}
a_\alpha &= -\frac{\mathscr{M}_\alpha^0}{\pi R_0^3E^*},\quad a_3 = -\frac{\mathscr{R}_3^0}{2\pi R_0}\left[2\alpha_2 + A_0\left(2\alpha_2 + \frac{\alpha_0}{R_0^2}\right)\right]^{-1},\\
k_0 &= -\frac{\mathscr{M}_3^0}{2\pi R_0}\left[\alpha_2R_0^2 + 2(\alpha_6+\alpha_7)\right]^{-1}.
\end{aligned} \tag{5.22}$$

In (5.21) and (5.22), the notations A_0, B_0, C_0 and A^*, B^*, C^* represent constants which are determined, respectively, by the systems of equations

$$\begin{aligned}
&(\alpha_1 + 2\alpha_2 + \alpha_0R_0^{-2})A_0 + \alpha_9B_0 + \beta_4C_0 = \alpha_1,\\
&\alpha_9A_0 + \alpha_4B_0 + \beta_5C_0 = \alpha_9,\\
&\beta_4A_0 + \beta_5B_0 + \beta_3C_0 = \beta_4,
\end{aligned} \tag{5.23}$$

and

$$\begin{aligned}
&\left(\alpha_1 + 2\alpha_2 + \frac{4\alpha_0\alpha_3}{\alpha_0 + \alpha_3 R_0^2}\right)A^* + \alpha_9 B^* + \beta_4 C^* = \alpha_1 + \frac{2\alpha_3\alpha_5}{\alpha_0 + \alpha_3 R_0^2},\\
&\alpha_9 A^* + (\alpha_4 + \alpha_8 R_0^{-2})B^* + (\beta_5 + \beta_2 R_0^{-2})C^* = \alpha_9,\\
&\beta_4 A^* + (\beta_5 + \beta_2 R_0^{-2})B^* + (\beta_3 + \beta_1 R_0^{-2})C^* = \beta_4,
\end{aligned} \tag{5.24}$$

while the constants D^* and E^* are specified by

$$\begin{aligned}
&D^* = \left[(\alpha_3 R_0^2 - \alpha_0)A^* + \alpha_5\right](\alpha_0 + \alpha_3 R_0^2)^{-1},\\
&E^* = \left(2\alpha_2 + \frac{\alpha_0}{R_0^2} - \frac{2\alpha_3\alpha_5}{\alpha_0 + \alpha_3 R_0^2}\right) + \left(2\alpha_2 - \frac{\alpha_5}{R_0^2} + \frac{4\alpha_0\alpha_3}{\alpha_0 + \alpha_3 R_0^2}\right)A^* - \frac{\alpha_5}{R_0^2}D^*.
\end{aligned} \tag{5.25}$$

We denote the solution (5.21) of the problem (P_1) by $v(a_i, k_0)$.

In what follows, we consider the flexure problem (characterized by $\mathscr{R}_3^0 = \mathscr{M}_i^0 = 0$) with respect to closed circular cylindrical shells. We search for a solution $v^0 \in K_{\text{II}}(\mathscr{R}_1^0, \mathscr{R}_2^0)$, $v^0 = \{\boldsymbol{u}^0, \boldsymbol{\delta}^0, \varphi^0\}$, such that v^0 is of the form (4.29).

By using the same method as in Section 4.2, adapted for closed circular cylindrical surfaces, we finally obtain the following solution of the problem (P_2):

$$\begin{aligned}
&u_\alpha^0 = \tfrac{1}{6}\epsilon_{\alpha\beta} b_\beta x_3^3 + A^* \epsilon_{\beta\gamma} b_\gamma x_\beta x_\alpha x_3, \quad u_3^0 = \epsilon_{\alpha\beta} b_\alpha x_\beta (\tfrac{1}{2}x_3^2 + K^*),\\
&\delta_\alpha^0 = R_0^{-1} b_\gamma (B^* \epsilon_{\beta\gamma} x_\beta x_\alpha - D^* \epsilon_{\alpha\beta} x_\beta x_\gamma) x_3, \quad \delta_3^0 = R_0^{-1} \epsilon_{\alpha\beta} b_\alpha x_\beta (\tfrac{1}{2}x_3^2 + L^*),\\
&\varphi^0 = C^* \epsilon_{\alpha\beta} b_\beta x_\alpha x_3.
\end{aligned} \tag{5.26}$$

The constants b_α are given in terms of $\mathscr{R}_\alpha^0$ by the relations

$$b_\alpha = -\frac{\epsilon_{\alpha\beta}\mathscr{R}_\beta^0}{\pi R_0^3 E^*}, \quad \alpha = 1, 2. \tag{5.27}$$

In (5.26) we have denoted by K^* and L^* the constant expressions

$$K^* = 2R_0^2\left[1 + A^* + \frac{\alpha_3}{\alpha_2}(A^* - D^*)\right], \quad L^* = R_0^2\left[\frac{\alpha_6 + \alpha_7}{\alpha_6 + \alpha_3 R_0^2}(1 + A^*) + D^*\right]. \tag{5.28}$$

6. Remarks and comments

6.1. Properties of the solutions

(i) We observe that the solutions v and v^0 determined in Sections 4 and 5 possess some of the characteristic properties of Saint-Venant's solution from the classical theory of elasticity.

Indeed, we see that (4.28) and (5.18) are the solutions of the problem (P_1) which satisfy the conditions (4.2). Also, we remark that v^0 given by (4.40) is the solution of the flexure problem which can be represented in the form (4.29). These representations of the solutions are analogous to those established by Ieşan (1987) for the classical Saint-Venant's solution.

We notice that the stress fields $\boldsymbol{N}$, $\boldsymbol{M}$ and h are independent of the axial coordinate z in the case of the extension-bending-torsion problem (see (4.8)), while for the flexure problem the stress fields $\boldsymbol{N}$, $\boldsymbol{M}$ and h depend linearly on z. This property corresponds to the characterization of Saint-Venant's solution given by Voigt (1887).

Clebsch (1862) has proved that Saint-Venant's solution can be distinguished among all the solutions of the relaxed problem by the property that the stress vector on any plane normal to the cross-sections of the cylinder is parallel to its generators.

Let us compute the contact force $\boldsymbol{N}$ acting on a straight line normal to the plane x_1Ox_2. Since $N_{ss} = V_s = 0$ in the case of open cylindrical surfaces, we have $\boldsymbol{N} = N_{sz}\boldsymbol{e}_3$. Hence, $\boldsymbol{N}$ is parallel to the generator and we find a property of the solution analogous to Clebsch's characterization.

In the case of the problem (P_1) for closed cylindrical shells, we obtain that $\boldsymbol{N} = B_\alpha \boldsymbol{e}_\alpha + N_{sz}\boldsymbol{e}_3$. Thus, the vector $\boldsymbol{N}$ is parallel to a fixed plane which is parallel to the generator and $\boldsymbol{N}$ has a constant projection on the plane x_1Ox_2.

(ii) From the results of Sections 4 and 5 we can deduce separately the solutions for the extension problem (characterized by $\mathscr{R}_\alpha^0 = \mathscr{M}_i^0 = 0$), the bending problem ($\mathscr{R}_i^0 = \mathscr{M}_3^0 = 0$) and the torsion problem ($\mathscr{R}_i^0 = \mathscr{M}_\alpha^0 = 0$).

In view of (4.24)–(4.27) and (5.14)–(5.17), we observe that the torsion deformation uncouples from the extension and bending of cylindrical Cosserat shells with voids. Moreover, for open cylindrical shells, extension and bending are also uncoupled.

Taking into account the expressions of the solutions obtained, we observe the interaction between the displacement fields and the porosity field. Indeed, the displacement $\boldsymbol{u}$ and the director displacement $\boldsymbol{\delta}$ are influenced by the porosity of the material, while the volume fraction field suffers changes due to the deformation of the shell.

In the case of the torsion problem we have $a_i = B_i = 0$. Hence, the solutions (4.28) and (5.18) reduce to the results obtained by Wenner (1968) for the torsion of cylindrical shells. Wenner (1968) has shown that these results are in agreement with the work of Reissner (1959), which deals with the classical theory of shells.

We remark that the torsion of cylindrical shells with voids has no effect on the porosity, since $\varphi = 0$.

(iii) The case of Cosserat shells made from an elastic material which is *not* porous is characterized by the relations

$$\varphi = 0, \quad \beta_k = 0 \quad (k = 1, \ldots, 5). \tag{6.1}$$

If we substitute (6.1) into the appropriate equations derived in this work, then we obtain the solution of the relaxed Saint-Venant's problem for Cosserat shells (*without* voids). Bîrsan (2004) has studied this problem in detail and has deduced its solution.

In what follows, we investigate the correspondence between the solution of Saint-Venant's problem in the theory of Cosserat shells made from an *elastic* material and the analogous results from the classical theory of shells.

6.2. Comparison with corresponding results from the classical shell theory

We consider cylindrical shells made from an homogeneous and isotropic *elastic* material, modelled as Cosserat surfaces. Since the shells are *not* porous, the volume fraction field is constant $\eta = 1$ and (6.1) holds. Let λ, μ be the Lamé constants, E is Young's modulus and ν designates Poisson's ratio for this elastic material. We denote by ζ the constant thickness of the shell and let

$$C = \frac{E\zeta}{1-\nu^2}, \quad D = \frac{E\zeta^3}{12(1-\nu^2)}.$$

In Naghdi (1972), Section 24, the constitutive coefficients $\alpha_1, \ldots, \alpha_9$ have been determined in terms of ν, C and D. Thus, we have

$$\alpha_1 = \alpha_9 = \frac{v(1-v)}{1-2v}C, \quad \alpha_2 = \frac{1-v}{2}C, \quad \alpha_4 = \frac{(1-v)^2}{1-2v}C, \\ \alpha_5 = vD, \quad \alpha_6 = \alpha_7 = \frac{1-v}{2}D, \tag{6.2}$$

while the coefficients α_3 and α_8 remain unspecified and have the orders of magnitude

$$\alpha_3 = O(C), \quad \alpha_8 = O(D). \tag{6.3}$$

In view of (6.1)–(6.3), we deduce that $\alpha_0 = D$, $\beta_0 = v$, $A = B = -v$. From the boundary-value problems (4.17) and (4.18), (5.6)–(5.8) we obtain

$$y_{(\alpha)} = -vx_\alpha + O(\zeta), \quad z_{(\alpha)} = -vx_\alpha + O(\zeta), \quad \alpha = 1,2 \tag{6.4}$$

and also (in the case of closed cylindrical shells)

$$F_{(\alpha)} = \frac{1}{C}(x'_\alpha + \epsilon_{\beta\alpha}x_\beta\sigma') + O(\zeta), \quad G_{(\alpha)} = -\frac{v}{(1-v)C}(x'_\alpha + \epsilon_{\beta\alpha}x_\beta\sigma') + O(\zeta), \\ F_{(3)} = -\frac{1}{C}\sigma' + O(\zeta), \quad G_{(3)} = \frac{v}{(1-v)C}\sigma' + O(\zeta), \tag{6.5}$$

while Γ, $\varphi_{(\alpha)}$ and $\Phi_{(i)}$ do not arise.

Remark. For a slightly less general theory of Cosserat shells (also discussed in Naghdi, 1972, Section 24) we have $\alpha_8 = 0$. Hence, $M^{\alpha 3}$ is absent. In this type of linear theory, which already includes the linear theories of shells currently employed in the literature, the relations (6.4) become $y_{(\alpha)} = z_{(\alpha)} = -v\ x_\alpha$ and (6.5) can be written in a simpler form, by dropping the terms $O(\zeta)$.

Introducing the formulae (6.1)–(6.5) into the appropriate expressions of the solutions derived in Sections 4 and 5, and neglecting some terms of orders $O(\zeta^2)$, $O(\zeta^3)$ or $O(\zeta^4)$, we obtain the approximate solution of Saint-Venant's problem presented below.

For the extension-bending-torsion problem, the displacement field is given by

$$u_\alpha = \frac{1}{2}\epsilon_{\alpha\beta}a_\beta(x_3^2 - vx_\gamma x_\gamma) - v(\epsilon_{\beta\gamma}a_\beta x_\gamma + a_3)x_\alpha - k_0\epsilon_{\alpha\beta}x_\beta x_3 + S_\alpha[a_i, B_i](s), \\ u_3 = (\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3)x_3 - k_0\int_0^s \epsilon_{\alpha\beta}x_\alpha x'_\beta \,\mathrm{d}s + \varepsilon k_0 \frac{2\mathscr{A}_c}{\bar{s}}s. \tag{6.6}$$

Here, ε is a parameter which is set to be $\varepsilon = 0$ for open cylindrical shells, whereas $\varepsilon = 1$ for closed cylindrical shells. Thus, by using the parameter ε, we are able to write the solution for both open and closed cylindrical shells in a single formula.

In (6.6) we have denoted by $S_\alpha[a_i,B_i](s)$ the function

$$S_\alpha[a_i, B_i](s) = \int_0^s \epsilon_{\beta\alpha}x'_\beta \int_0^\xi \left[v(\epsilon_{\gamma\delta}a_\gamma x_\delta + a_3)\sigma' + \frac{1}{D}(\epsilon_{\gamma\delta}B_\gamma x_\delta + B_3)\right]\mathrm{d}\tau\,\mathrm{d}\xi, \tag{6.7}$$

where a_i and B_i are constants and $s \in [0, \bar{s}]$.

The constants k_0 and $a_i (i = 1,2,3)$ can be expressed in terms of the resultant axial force $\mathscr{R}_3^0$, bending moments $\mathscr{M}_\alpha^0$ and twisting moment $\mathscr{M}_3^0$ acting on the end edges of the cylindrical shell, by the relations

$$a_3 = -\frac{\mathscr{R}_3^0}{E\zeta\bar{s}}, \tag{6.8}$$

$$\epsilon_{\beta\gamma}a_\beta \int_0^{\bar{s}} x_\alpha x_\gamma \,\mathrm{d}s = \epsilon_{\alpha\beta}\frac{\mathscr{M}_\beta^0}{E\zeta}, \tag{6.9}$$

$$k_0 = -\frac{\mathscr{M}_3^0}{\mu \widehat{D}}. \tag{6.10}$$

In (6.10), $\widehat{D}$ denotes the constant

$$\begin{aligned} &\widehat{D} = \frac{1}{3}\zeta^3 \bar{s} \quad \text{for open cylindrical shells,} \\ &\widehat{D} = 4\zeta \frac{1}{\bar{s}} \mathscr{A}_c^2 \quad \text{for closed cylindrical shells.} \end{aligned} \tag{6.11}$$

As we know from Section 4 , we have $B_i = 0 (i = 1, 2, 3)$ for open cylindrical shells. On the other hand, for closed cylindrical shells the constants B_i can be determined in terms of $a_i (i = 1, 2, 3)$ from the relations

$$\begin{aligned} &\epsilon_{\beta\gamma}\frac{B_\beta}{D}\int_0^{\bar{s}} x_\alpha x_\gamma \,\mathrm{d}s = -\nu \int_0^{\bar{s}} (\epsilon_{\beta\gamma} a_\beta x_\gamma + a_3) x_\alpha \sigma' \,\mathrm{d}s, \\ &\frac{B_3}{D}\bar{s} = -\nu \int_0^{\bar{s}} (\epsilon_{\beta\gamma} a_\beta x_\gamma + a_3)\sigma' \,\mathrm{d}s. \end{aligned} \tag{6.12}$$

(These equations are equivalent to the conditions $S_\alpha[a_i, B_i](\bar{s}) = S'_\alpha[a_i, B_i](\bar{s}) = 0$.)

We observe that the constants a_3, a_α and k_0, respectively, represent measures of stretch, curvature and twist of the cylindrical shell considered as a beam. Thus, the relations (6.8)–(6.11) *coincide* with the classical results given in Reissner and Tsai (1972) (for the case of an *isotropic* material). The same results can be obtained from the work of Berdichevsky et al. (1992) for the case of isotropic thin-walled closed-cross-section tubes (see also Ladevèze et al., 2004). We mention that the torque-twist relations (6.10) and (6.11) are established in Timoshenko and Goodier (1951), Sections 94, 98, 102 and in Sokolnikoff (1956), Section 47. The problem of Saint-Venant torsion of thin-walled tubes is also discussed in Reissner (1970).

The displacement in the axial direction u_3 given by $(6.6)_2$ coincide with the results obtained from the work of Reissner and Tsai (1972) for isotropic materials. Also, the components u_α of the displacement vector for open cylindrical shells have the same form as the solution found by Reissner and Tsai, except for the term $S_\alpha[a_i,B_i](s)$ which appears in $(6.6)_1$. As we will show later, the term $S_\alpha[a_i,B_i](s)$ vanishes for *initially flat* shells (i.e. *plates*) and for *circular* closed cylindrical shells, so that the displacements u_α correspond exactly to the results of Reissner and Tsai for these cases.

In the same way, using the relations (6.1)–(6.5) and the results of Sections 4 and 5 we find that the approximate solution of the flexure problem is

$$\begin{aligned} u_\alpha^0 &= \tfrac{1}{2}\epsilon_{\alpha\beta} b_\beta x_3 \left(\tfrac{1}{3}x_3^2 - \nu x_\gamma x_\gamma\right) - \nu(\epsilon_{\beta\gamma} b_\beta x_\gamma) x_\alpha x_3 - k_2 \epsilon_{\alpha\beta} x_\beta x_3 + x_3 S_\alpha[b_\beta, 0, C_i](s), \\ u_3^0 &= \tfrac{1}{2}(\epsilon_{\alpha\beta} b_\alpha x_\beta) x_3^2 + \tfrac{1}{2}\nu x_1^2 \left(b_1 x_2 - \tfrac{1}{3} b_2 x_1\right) + \tfrac{1}{2}\nu x_2^2 \left(\tfrac{1}{3} b_1 x_2 - b_2 x_1\right) \\ &\quad - k_2 \int_0^s \epsilon_{\alpha\beta} x_\alpha x'_\beta \,\mathrm{d}s - \nu \int_0^s (b_1 x_1^2 x'_2 - b_2 x_2^2 x'_1)\,\mathrm{d}s - 2(1+\nu)\int_0^s \int_0^\xi (\epsilon_{\alpha\beta} b_\alpha x_\beta)\,\mathrm{d}\tau \,\mathrm{d}\xi \\ &\quad - \varepsilon \frac{1+\nu}{\mathscr{A}_c} s \int_0^{\bar{s}} (\epsilon_{\alpha\beta} x'_\alpha x_\beta) \int_0^s \epsilon_{\gamma\delta} b_\gamma x_\delta d\xi \,\mathrm{d}s + \int_0^s x'_\alpha S_\alpha[b_\beta, 0, C_i]\,\mathrm{d}s. \end{aligned} \tag{6.13}$$

The constants $b_\alpha (\alpha = 1,2)$ can be expressed in terms of $\mathscr{R}_\alpha^0$ from the system of algebraic equations

$$\epsilon_{\beta\gamma} b_\beta \int_0^{\bar{s}} x_\alpha x_\gamma \,\mathrm{d}s = -\frac{\mathscr{R}_\alpha^0}{E\zeta}, \tag{6.14}$$

while the constant k_2 which appear in (6.13) has the following value:

$$k_2 = \frac{E\zeta}{\mu\widehat{D}}\int_0^{\bar{s}}(\epsilon_{\alpha\beta}x'_\alpha x_\beta)\int_0^s \epsilon_{\gamma\delta}b_\gamma x_\delta\,\mathrm{d}\xi\,\mathrm{d}s - \frac{\varepsilon}{2\mathscr{A}_c}\left[v\int_0^{\bar{s}}(b_1x_1^2x_2' - b_2x_2^2x_1')\,\mathrm{d}s + 2(1+v)\int_0^{\bar{s}}\int_0^s(\epsilon_{\alpha\beta}b_\alpha x_\beta)\,\mathrm{d}\xi\,\mathrm{d}s - \int_0^{\bar{s}}x'_\alpha S_\alpha[b_\beta,0,C_i]\,\mathrm{d}s\right]. \qquad (6.15)$$

The constants $C_i (i = 1, 2, 3)$ are set to be $C_i = 0$ for open cylindrical shells, whereas for closed cylindrical shells C_i are given in terms of b_α by the relations

$$\epsilon_{\beta\gamma}\frac{C_\beta}{D}\int_0^{\bar{s}}x_\alpha x_\gamma\,\mathrm{d}s = -v\int_0^{\bar{s}}(\epsilon_{\beta\gamma}b_\beta x_\gamma)x_\alpha\sigma'\,\mathrm{d}s,$$
$$\frac{C_3}{D}\bar{s} = -v\int_0^{\bar{s}}(\epsilon_{\beta\gamma}b_\beta x_\gamma)\sigma'\,\mathrm{d}s. \qquad (6.16)$$

In what follows, we particularize the above results for the cases of *initially flat* cylindrical shells (i.e. rectangular plates) and *circular* closed cylindrical shells.

Consider first the deformation of rectangular plates, as a special case of the problem solved in Section 4, with

$$x_1(s) = s - l_0, \quad x_2(s) = 0, \quad s \in [0, 2l_0]. \qquad (6.17)$$

If we substitute (6.17) into (6.6), then we obtain the following solution of the extension-bending-torsion problem:

$$u_1 = \frac{1}{2}a_2(x_3^2 + vx_1^2) - va_3x_1,$$
$$u_2 = -\frac{1}{2}a_1(x_3^2 - vx_1^2) + k_0x_1x_3, \qquad (6.18)$$
$$u_3 = -a_2x_1x_3 + a_3x_3.$$

In the case of plates, the constants a_α can no longer be computed from the approximate relations (6.9), since the determinant of the system is null. Instead, by using directly (4.25) and (4.26) we deduce

$$I_{11} = -\frac{1}{6}l_0E\zeta^3, \quad I_{22} = -\frac{2}{3}l_0^3E\zeta, \quad I_{12} = I_{21} = 0,$$

so that

$$a_1 = -\frac{6\mathscr{M}_1^0}{l_0E\zeta^3}, \quad a_2 = -\frac{3\mathscr{M}_2^0}{2l_0^3E\zeta}. \qquad (6.19)$$

From (6.8) and (6.10) we get

$$a_3 = -\frac{\mathscr{R}_3^0}{2l_0E\zeta}, \quad k_0 = -\frac{3\mathscr{M}_3^0}{2l_0\mu\zeta^3}. \qquad (6.20)$$

The solution (6.18)–(6.20) coincides with the result presented by Reissner and Tsai (1972) for plates, when the elastic material is isotropic.

In view of (6.13) and (6.17), for the flexure problem we obtain the following displacement field:

$$u_1^0 = \tfrac{1}{2}b_2x_3(\tfrac{1}{3}x_3^2 + vx_1^2),$$
$$u_2^0 = -\tfrac{1}{2}b_1x_3(\tfrac{1}{3}x_3^2 - vx_1^2), \qquad (6.21)$$
$$u_3^0 = \tfrac{2+v}{6}b_2x_1^3 - \tfrac{1}{2}b_2x_1x_3^2 - (1+v)b_2l_0^2x_1.$$

Here, b_α are given by

$$b_1 = -\frac{6\mathscr{R}_2^0}{l_0 E\zeta^3}, \quad b_2 = -\frac{3\mathscr{R}_1^0}{2l_0^3 E\zeta}. \tag{6.22}$$

The flexure problem for plates has also been studied by Timoshenko and Goodier (1951), Section 20, in the case when $\mathscr{R}_1^0 \neq 0, \mathscr{R}_2^0 = 0$, and the result is the same as in (6.21) and (6.22).

Let us consider now Saint-Venant's problem for circular closed cylindrical shells. In this case, the functions $x_\alpha(s)$ are given by (5.19) and we notice that $S_\alpha[a_i, B_i](s) = S_\alpha[b_\beta, 0, C_i](s) = 0$. Then, from (6.6) and (6.8)–(6.11) we deduce the solution of the extension-bending-torsion problem

$$\begin{aligned} u_\alpha &= \frac{1}{2}\epsilon_{\alpha\beta}a_\beta x_3^2 - v(\epsilon_{\beta\gamma}a_\beta x_\gamma + a_3)x_\alpha - k_0\epsilon_{\alpha\beta}x_\beta x_3, \\ u_3 &= (\epsilon_{\alpha\beta}a_\alpha x_\beta + a_3)x_3, \end{aligned} \tag{6.23}$$

where

$$a_3 = -\frac{\mathscr{R}_3^0}{2\pi R_0 E\zeta}, \quad a_\alpha = -\frac{\mathscr{M}_\alpha^0}{\pi R_0^3 E\zeta}, \quad k_0 = -\frac{\mathscr{M}_3^0}{2\pi R_0^3 \mu\zeta}. \tag{6.24}$$

The same solution (6.23) and (6.24) is obtained from the results of Reissner and Tsai (1972).

From (6.13) and (6.14) we deduce the following solution of the flexure problem for circular closed cylindrical shells:

$$\begin{aligned} u_\alpha^0 &= \tfrac{1}{6}\epsilon_{\alpha\beta}b_\beta x_3^3 - v(\epsilon_{\beta\gamma}b_\beta x_\gamma)x_\alpha x_3, \\ u_3^0 &= (\epsilon_{\alpha\beta}b_\alpha x_\beta)[\tfrac{1}{2}x_3^2 + 2(1+v)R_0^2], \end{aligned} \tag{6.25}$$

where

$$b_\alpha = -\frac{\epsilon_{\alpha\beta}\mathscr{R}_\beta^0}{\pi R_0^3 E\zeta}. \tag{6.26}$$

Remark. The solution (6.23)–(6.26) of Saint-Venant's problem for circular cylindrical shells can also be obtained directly from the results of Section 5.2, if we use relations (6.1)–(6.3) and then make some approximations, by neglecting terms of order $O(\zeta^2)$.

We close this section by observing the analogy between the solution (6.6)–(6.13) for cylindrical shells and the classical Saint-Venant's solution for *solid* cylinders.

In order to make this analogy more visible, we adopt the notations

$$\bar{a}_\alpha = \epsilon_{\beta\alpha}a_\beta, \quad \bar{b}_\alpha = \epsilon_{\beta\alpha}b_\beta,$$

and introduce the functions

$$\begin{aligned} \bar{\varphi}(s) &= -\int_0^s \epsilon_{\alpha\beta}x_\alpha x'_\beta \, \mathrm{d}s + \varepsilon\frac{2\mathscr{A}_c}{\bar{s}}s, \\ \bar{\psi}(s) &= -v\left[\int_0^s (\bar{b}_1 x'_1 x_2^2 + \bar{b}_2 x'_2 x_1^2)\,\mathrm{d}s - \varepsilon\frac{s}{\bar{s}}\int_0^{\bar{s}} (\bar{b}_1 x'_1 x_2^2 + \bar{b}_2 x'_2 x_1^2)\,\mathrm{d}s\right] \\ &\quad - 2(1+v)\left(\int_0^s \int_0^\xi \bar{b}_\alpha x_\alpha \mathrm{d}\tau \mathrm{d}\xi - \varepsilon\frac{s}{\bar{s}}\int_0^{\bar{s}}\int_0^\xi \bar{b}_\alpha x_\alpha \mathrm{d}\tau\mathrm{d}\xi\right). \end{aligned}$$

Then, the solution (6.6)–(6.13) of Saint-Venant's problem can be written in the form

$$\begin{aligned} u_\alpha &= -\tfrac{1}{2}\bar{b}_\alpha x_3\left(\tfrac{1}{3}x_3^2 - v x_\beta x_\beta\right) - v\bar{b}_\beta x_\beta x_\alpha x_3 - \tfrac{1}{2}\bar{a}_\alpha\left(x_3^2 - v x_\beta x_\beta\right) \\ &\quad - v(\bar{a}_\beta x_\beta + a_3)x_\alpha - (k_0 + k_2)\epsilon_{\alpha\beta}x_\beta x_3 + S_\alpha[a_i, B_i] + x_3 S_\alpha[b_\beta, 0, C_i], \\ u_3 &= \tfrac{1}{2}\bar{b}_\alpha x_\alpha x_3^2 + \tfrac{1}{2}v x_1^2\left(\tfrac{1}{3}\bar{b}_1 x_1 + \bar{b}_2 x_2\right) + \tfrac{1}{2}v x_2^2\left(\bar{b}_1 x_1 + \tfrac{1}{3}\bar{b}_2 x_2\right) \\ &\quad + (\bar{a}_\alpha x_\alpha + a_3)x_3 + (k_0 + k_2)\bar{\varphi} + \bar{\psi} + \int_0^s x'_\alpha S_\alpha[b_\beta, 0, C_i]\,\mathrm{d}s \\ &\quad - \varepsilon\frac{s}{\bar{s}}\int_0^{\bar{s}} x'_\alpha S_\alpha[b_\beta, 0, C_i]\,\mathrm{d}s. \end{aligned} \tag{6.27}$$

Except for the terms containing S_α , the displacement field (6.27) has the same form as the well-known Saint-Venant's solution for solid cylinders. The function $\bar{\varphi}(s)$ represents the torsion function associated to the deformation of cylindrical shells.

The analogy with the classical Saint-Venant's solution can be extended also to the relations between the constants $\bar{a}_\alpha, a_3, \bar{b}_\alpha, k_0$ appearing in (6.27) and the resultant forces and moments $\mathscr{R}_i^0, \mathscr{M}_i^0$. Without loss of generality, we can choose the Cartesian coordinate system such that the origin is at the centroid, with the coordinate axes being principal axes, i.e. we assume that (4.1) holds and that $\int_0^{\bar{s}} x_1 x_2\,\mathrm{d}s = 0$. Then, the relations (6.8)–(6.10) and (6.14) become

$$\begin{aligned} &\bar{a}_1 E\left(\zeta\int_0^{\bar{s}} x_1^2\,\mathrm{d}s\right) = \mathscr{M}_2^0, \quad \bar{a}_2 E\left(\zeta\int_0^{\bar{s}} x_2^2\,\mathrm{d}s\right) = -\mathscr{M}_1^0, \\ &a_3 E(\zeta\bar{s}) = -\mathscr{R}_3^0, \quad k_0(\mu\widehat{D}) = -\mathscr{M}_3^0, \\ &\bar{b}_1 E\left(\zeta\int_0^{\bar{s}} x_1^2\,\mathrm{d}s\right) = -\mathscr{R}_1^0, \quad \bar{b}_2 E\left(\zeta\int_0^{\bar{s}} x_2^2\,\mathrm{d}s\right) = -\mathscr{R}_2^0. \end{aligned} \tag{6.28}$$

These relations are analogue with the well-known Saint-Venant's formulae for bending, extension, torsion and flexure, respectively. Indeed, in (6.28) the product $\zeta\bar{s}$ corresponds to the area of the cross-section Σ of the three-dimensional cylindrical shell, whereas the expression $\zeta\int_0^{\bar{s}} x_\alpha^2\,\mathrm{d}s$ is the analogue of $\int_\Sigma x_\alpha^2\,\mathrm{d}s(\alpha = 1, 2)$. The quantity $\mu\widehat{D}$ represents the torsional rigidity of the (open or closed) cylindrical shell.

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